A Robust Approach to Risk Aversion

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PRELIMINARY AND INCOMPLETE

Abstract

We explore the whole set of Kreps and Porteus recursive utility functions and look at classes of utility functions that are well ordered in terms of risk aversion. It is found that the only possibility is provided by the class of preferences introduced by Hansen and Sargent in their robustness analysis. The paper suggests therefore a shift from the traditional approach to study risk aversion in recursive problems. Applications show that working with these preferences leads to unambiguous and intuitive result on the impact of risk aversion on the risk free rate, the market price for risk and risk sharing in general equilibrium.

Keywords: risk aversion, recursive utility, robustness, risk free rate, equity premium, risk sharing.

JEL codes: E2, E43, E44.

1 Introduction

Since Koopmans (1960) early article, the assumption of preference stationarity plays a central role in the modeling of intertemporal choice under uncertainty. For a number of problems it makes indeed sense to assume that the agent's objective should be independent of what happened in the past and should have a structure that is independent of the calendar year in which the agent is living. Preference stationarity is then required for such preferences to generate time consistent planning.

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The economics abounds of work that focus on stationary preferences. In Decision Theory, Epstein (1983), Epstein and Zin (1989), Klibanoff, Marinacci, and Mukerji (2009) are famous articles who extended Koopmans's initial contribution to more general settings. The assumption of preference stationarity is even more pregnant in applied works, as it leads to problems with a recursive structure that can be tackled using dynamic programming methods. A great part of the macro-economic literature relies on models assuming stationary preferences.

Imposing preference stationarity makes it however difficult to discuss the role of risk aversion. When considering infinite horizon setting, it is indeed impossible to have stationary preferences that fit in the expected utility framework and are comparable in terms of risk aversion. Discussing the role of risk aversion -while maintaining the assumption of preference stationarity- involves then either departing form the expected utility framework, as in Epstein and Zin (1989) or considering preferences defined on smaller domains, as in Bommier (2012)[1].

The most popular option consists in using Epstein and Zin's (1989) isoelastic preferences which extends Kreps and Porteus (1978) to a stationary setting. These preferences were found to be extremely useful for solving a lot of problems, having the big advantage to be easily tractable. Relying on Epstein and Zin's (1989) isoelastic preferences is now considered as the standard procedure to study the role of risk aversion in intertemporal problems. Though, such an approach has several caveats. First, as it was emphasized in Chew and Epstein (1990), the preferences introduced in Epstein and Zin (1989) generally fail to fulfill an intuitive notion of ordinal dominance. An agent having preferences of this kind may end choosing lotteries that are first-order stochastically dominated by other available lotteries. In that sense, these preferences do not conform with a natural assumption of preference monotonicity. This has a number of unpleasant consequences. In particular, as was shown in a two period setting by Bommier, Chassagnon and Le Grand (2010), Epstein and Zin isoelastic preferences are not well ordered in terms to risk aversion, which lead to misleading conclusion about the role of risk aversion on standard problems such as that of precautionary savings.

The object of this paper is to explore wether one could find other classes of stationary preferences that allow to disentangle ordinal and risk preferences, without having the caveats mentioned above. We will actually explore the whole set of recursive preferences that are consistent with Kreps and Porteus (1978) framework and look for classes of preferences that fulfill ordinal dominance and can be used to study of risk aversion. Our main result shows that there remains only one possibility.

[1] Bommier (2012) explores the case where consumption paths that have to take a given value after a finite amount of time. It is then possible to have classes of preferences which fit in the expected utility framework, are well-ordered in terms of risk aversion and are stationary.

It involves using preferences given by the following recursion:

$$\begin{aligned} U_{t+1} &= (1-\beta)u(c_t) - \tfrac{\beta}{k}\log(E\left[e^{-kU_t}\right]) \text{ if } k \neq 0 \\ U_{t+1} &= (1-\beta)u(c_t) + \beta E[U_t] \text{ if } k = 0 \end{aligned} \quad (1)$$

The parameter k is then determining the degree of risk aversion, a larger k being associated with stronger risk aversion, even for the strong notion or comparative risk aversion that was introduced in Bommier, Chassagnon and Le Grand (2010). Interestingly enough, this class of preferences corresponds to the one introduced by Hansen and Sargent (1995), in their approach to robustness analysis. As was noted by Tallarini (2000) and Hansen and Sargent (2007) such a class intersect with the standard Epstein and Zin's isoelastic preferences in the particular case where u is a log function (therefore when an intertemporal elasticity of substitution of one is assumed) but are of different nature otherwise. Also, in the case where β equals zero, and preferences defined on a restricted domain in order to avoid convergence problems, these preferences correspond the multiplicative model discussed in as in Bommier (2012).

In order to illustrate why the literature would gain in using specifications which are well ordered in terms of risk aversion, we develop two different applications. In a first one, we investigate the impact of risk aversion on the risk-free rate and the market price of risk in a simple endowment economy. Relying on preferences defined by the recursion (1), it is found that more risk aversion is necessarily associated with a lower risk-free rate and a larger market price for risk. Such finding have clear-cut consequences for discussing key issues such as the choice of proper social discount rate to evaluate public policy. In particular this indicates that, under fairly general conditions, the social discount rate should be a decreasing function of the planner's risk aversion.

The second application bears on risk sharing in a closed economy. We will consider two agents who face two sources of risk (a productivity risk and an investment risk) and which interact through a standard market - both agents acting as price takers. It is shown that at the equilibrium the less risk averse agent is the one that end up taking less risk, and that the risk premium is increasing with average risk aversion but, for a given average, decreasing with heterogeneity in risk aversion.

The remaining of the paper is organized as follows. In the following section we expose the setting. Section 3 formalizes the notions of ordinal dominance and provide a representation result about recursive preferences fulfilling ordinal dominance. Section 4 discusses notion of comparative risk aversion and show that the preferences that we obtained in Section 3 are well ordered in terms of risk aversion. Section 5 deals with applications, exploring the impact of risk aversion in an endowment economy and risk sharing in general equilibrium. Section 6 discusses further some properties of robust preferences, while Section 7 concludes.

2 The setting

2.1 Preference domain

We consider preferences defined over the set of temporal lotteries in an infinite horizon setting. For simplicity sake we will assume that per-period consumption is bounded. We denote by $\underline{c}$ and $\overline{c}$ the lowest and highest possible values of consumption. The set of temporal lotteries can be defined as in Epstein and Zin (1989), which extended Kreps and Porteus (1978) to a recursive setting. The construction of such a set is in fact technically complex and the object of the whole Section 2 and Appendix 1 in Epstein and Zin (1989). It needs not to be replicated in the current paper. We shall precisely consider the set that is denoted by D in Epstein and Zin and defined by equation (2.10) of their article, with the additional assumption however that each period consumption as to lie in $[\underline{c}, \overline{c}]$.

An element of D is generally noted (c, m), where $c \in [\underline{c}, \overline{c}]$ is instantaneous consumption and m is a as an element of $M(D)$, that is a probability measure over the set D. An element (c, m) of D can therefore be interpreted the combination of a deterministic first period consumption and a lottery on future consumption programs.

There are some subsets of D that will appear in several instances in the paper and to which we given specific names. We denote by $D_0 \subset D$ the set of degenerate temporal lotteries. This set is isomorph to $[\underline{c}, \overline{c}]^{\mathbb{N}}$, the set of admissible consumption paths. The set D_0 includes in particular degenerate lotteries that give a constant consumption path. For any $c \in [\underline{c}, \overline{c}]$ we will denote by $c_\infty \in D_0$ the degenerate lottery that pays c for sure in all period of times.

Recursively, for all $n > 0$ we define $D_n \supset D_{n-1}$ by

$$(c, m) \in D_n \Leftrightarrow m \in M(D_{n-1})$$

The set $D_n \subset D$ is the set of lotteries that resolve in at most n periods. It was proven in Epstein and Zin (1989) that $\bigcup_{n \in \mathbb{N}} D_n$ is dense into D. As a consequence, whenever we will introduce concepts that are defined on $\bigcup_{n \in \mathbb{N}} D_n$ one could easily extend these concepts to D.

2.2 Recursive Kreps and Porteus preferences

Our paper explores the set of recursive utility functions that fit in the framework introduced by Kreps and Porteus. We will restrict our assumption to monotonic preferences, implying therefore that $\underline{c}_\infty$ and $\overline{c}_\infty$ are the elements of D that provide the lowest and highest levels of utility. By utility normalization, there is no loss of generality to assume that a utility function U defined on D has to fulfill $U(\underline{c}_\infty) = 0$ and $U(\overline{c}_\infty) = 1$. That lead us to provide the following formal definition:

Definition 1 *A utility function* $U : D \to [0,1]$*, such that* $U(\underline{c}_\infty) = 0$ *and* $U(\overline{c}_\infty) = 1$ *is said to be KP-recursive if and only if it there exists a function* $W : [\underline{c}, \overline{c}] \times [0,1] \to [0,1]$ *which is twice continuously differentiable and increasing in both arguments such that for all* $c_0 \in \mathbb{R}_+$ *and* $m \in M(D)$:

$$U(c_0, m) = W(c_0, E[U([m])]) \tag{2}$$

where $U([m])$ *denotes the probability measure of utility implied by* m.

The function W *will be called an admissible aggregator for the KP-recursive utility function.*

Moreover, a preference relation on D *will be said to be KP-recursive if it can be represented by a KP-recursive utility function.*

The most common example of KP-recursive utility function is the case of **additive separable utility**.

$$U(c_0, m) = (1-\beta)E_m\left[\sum_{i=0}^{+\infty} \beta^i u(c_i)\right]$$

where u is an increasing function such that $u(\underline{c}) = 0$ and $u(\overline{c}) = 1$. For this additive separable specification equation (2) holds when:

$$W(x, y) = (1-\beta)u(x) + \beta y \tag{3}$$

Another famous example of KP-recursive preferences is the case of **Epstein-Zin isoelastic preferences**, usually represented by utility functions fulfilling the recursion (see Epstein and Zin (2001) for example):

$$V(c,m) = \left[(1-\beta)c^\rho + \beta(E_t[V([m])^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} \text{ with } 1 < \rho \neq 0$$

or

$$V(c,m) = \exp\left((1-\beta)\log(c_t) + \frac{\beta}{\alpha}\log(E_t[V([m])^\alpha])\right) \text{ with } \rho = 0$$

The utility function $V(c,m)$ is not KP-recursive in the sense of Definition 1. But, that is just a matter of normalization since an equivalent representation of the preferences obtained by choosing:

$$U(c,m) = \frac{V(c,m)^\alpha - \underline{c}^\alpha}{\overline{c}^\alpha - \underline{c}^\alpha}$$

does fulfill the requirements of Definition 1 with:

$$W(x,y) = \frac{\left((1-\beta)x^\rho + \beta\left[y\left(\overline{c}^\alpha - \underline{c}^\alpha\right) + \underline{c}^\alpha\right]^{\frac{\rho}{\alpha}}\right)^{\frac{\alpha}{\rho}} - \underline{c}^\alpha}{\overline{c}^\alpha - \underline{c}^\alpha} \text{ when } 1 < \rho \neq 0$$

or

$$W(x,y) = \frac{\exp\left(\alpha(1-\beta)\log(c) + \beta\log(y\left(\overline{c}^\alpha - \underline{c}^\alpha\right) + \underline{c}^\alpha)\right) - \underline{c}^\alpha}{\overline{c}^\alpha - \underline{c}^\alpha} \text{ when } \rho = 0$$

The two examples we provided so far (standard additive expected utility model and Epstein and Zin isoleastic specifications) are just two particular cases among many other possibilities. By far, there are the most widely used in economics, though they are inappropriate to study the role of risk aversion[2].

Our purpose in the current paper is to look for classes of KP-recursive preferences that fulfill ordinal dominance and suitable to discuss the role of risk aversion. This will lead to preferences that can be represented by $V(c,m)$ such that

$$\begin{array}{rcl} V(c,m) & = & (1-\beta)u(c) - \frac{\beta}{k}\log(E\left[e^{-kV([m])}\right]) \text{ if } k \neq 0 \\ V(c,m) & = & (1-\beta)u(x) + \beta E[V([m])] \text{ if } k = 0 \end{array} \tag{4}$$

for some function u. Even though their motivation was different from ours, this specification was first used by Hansen and Sargent in their work on robustness. For this reason we shall call them **robust preferences**. Here again, the utility representation given in (4) does not fulfill the requirement of Definition 1. But posing

$$\begin{array}{rcl} U(c,m) & = & \dfrac{e^{-kV(c,m)} - e^{-ku(\underline{c})}}{e^{-ku(\overline{c})} - e^{-ku(\underline{c})}} \text{ if } k \neq 0 \\ U(c,m) & = & \dfrac{V(c,m) - u(\underline{c})}{u(\overline{c}) - u(\underline{c})} \text{ if } k = 0 \end{array}$$

we obtain a utility function $U(c,m)$ which represent the same preferences and fulfills the requirements of Definition 1 with an aggregator given by:

$$W(x,y) = \frac{e^{-k(1-\beta)(u(x)-u(\underline{c}))}\left[1 + y(e^{-k(u(\overline{c})-u(\underline{c}))} - 1)\right]^{\beta} - 1}{e^{-k(u(\overline{c})-u(\underline{c}))} - 1} \text{ when } k \neq 0 \tag{5}$$

$$W(x,y) = (1-\beta)\frac{u(x) - u(\underline{c})}{u(\overline{c}) - u(\underline{c})} + \beta y \text{ when } k = 0 \tag{6}$$

It should be noticed moreover that when taking $u(c) = \log(c)$ in the above equation we obtain the same aggregator as with Epstein and Zin representation with $\rho = 0$. This emphasizes that in the particular case where the intertemporal elasticity of substitution is take equal to one, then Epstein and Zin representation also belongs to the class of robust preferences, as was noticed by Tallarini (2000) for example. But, it is also clear that it is not true whenever $\rho \neq 0$. Last, we shall remark that when u is chosen such that $u(\underline{c}) = 0$ and $u(\overline{c}) = 1$, the above equations can be rewritten as:

$$\begin{array}{rcl} 1 - \widetilde{k}W(x,y) & = & e^{-k(1-\beta)u(x)}\left[1 - \widetilde{k}y\right]^{\beta} \text{ when } k \neq 0, \text{ with } \widetilde{k} = 1 - e^{-k} \\ W(x,y) & = & (1-\beta)u(x) + \beta y \text{ when } k = 0 \end{array}$$

which looks somewhat nicer to read.

We shall now introduce the notion of ordinal dominance and show that this leads to restrict our attention to robust preferences.

[2]See Bommier, Chassagnon and Legrand (2010).

3 Recursive preferences fulfilling Ordinal Dominance

3.1 Definition of Ordinal Dominance

The basic idea of ordinal dominance is that an individual should not prefer a lottery that is stochastically dominated at the first order by another one. When considering "atemporal uncertainty setting" -in the sense that uncertainty is always resolved with the same timing- the notion of ordinal dominance is relatively standard and can be found for example in Chew and Epstein (1990). Though, the extension to "temporal uncertainty setting" -where preferences for the timing may make sense- still has to be worked out. Hayashi and Miao (2011), who discuss ambiguity aversion in a temporal setting, suggest to use a notion of dominance that depends on the agent preferences (see their Axiom B7). When proceeding in such a way, first order stochastic dominance is implicitly defined as a partial order that depends on individual preferences under uncertainty. Such an approach is relevant to impose some internal consistency of agents preferences under uncertainty. It is explicitly used in that way by Hayashi and Miao (2011), who relate it to another assumption of preference consistency (their Axiom A7). Though, it does not respond to our wish to have a concept of ordinal dominance that is independent of agent risk preferences and can be used to discuss preference monotonicity and comparative risk aversion.

In line with Chew and Epstein (1990), we shall introduce a relation of stochastic dominance that only depends the relation of preference over deterministic objects. Therefore two agents with the same ordinal preferences will agree on whether a temporal lottery stochastically dominates another one. This will turn to be essential for comparing risk aversion. Though, by considering that the notion of stochastic dominance may depend on ordinal preferences, we readily implicitly admit that risk aversion is only comparable among agents having the same ranking over deterministic consumption paths. In that respect we follow the most of the literature on comparative risk (and ambiguity) aversion, including Kilhstrom and Mirman (1974) and Epstein and Zin (1989), Chew and Epstein (1990).

In order to define first order stochastic dominance over temporal lotteries, we proceed recursively to extend the approach used by Chew and Epstein (1990). Formally let us assume that D_0 is provided with a preference relation $\succeq_0$.

Given that relation of preferences on D_0 we define a notion of ordinal dominance FSD_1 on D_1 by:

$$
\begin{aligned}
(c,m)\ FSD_1\ (c',m') \quad &\Leftrightarrow \\
\text{for all } x \quad &\in \quad D_0 \text{ we have } m\left(\{x'|(c,x') \succeq_0 (c,x)\right) \leq m'\left(\{x'|\,(c',x') \succeq_0 (c',x)\}\right)
\end{aligned}
$$

This definition of FSD_1 corresponds to that of Chew and Epstein (1990). Comparison of lotteries that resolved in more than 1 period was not suggested by Chew and Epstein (1990) since they focus on cases where uncertainty is always resolved after period 1. However this can then be done recursively, as follows. Given FSD_n a relation of first order stochastic dominance on D_n, we define a relation of first order stochastic dominance FSD_{n+1} on D_{n+1} by:

$$\begin{aligned}(c,m)FSD_n(c',m') \quad &\Leftrightarrow \\ \text{for all } x \quad &\in \quad D_{n-1} \text{ we have } m\left(\{x'|(c,x')\ FSD_{n-1}\ (c,x)\}\right) \leq m'\left(\{x'|(c',x')\ FSD_{n-1}\ (c',x)\}\right)\end{aligned}$$

Note that if x and $y \in D_n$ and $xFSD_ny$ then $xFSD_{n+1}y$. The relation FSD_{n+1} which allows to compare temporal lotteries that resolve in at most $n+1$ periods of time is therefore consistent with FSD_n which compares lotteries resolving in at most n periods of time.

When proceeding in such a way, first order stochastic dominance is only defined over lotteries that resolve in a finite amount of time, but as $\bigcup_{n\in\mathbb{N}} D_n$ is dense into D, this is sufficient to define strong enough notion of ordinal dominance:

Definition 2 *(Ordinal dominance) A relation of preferences $\succeq$ on D is said to be fulfill ordinal dominance, if for all n and all (c,m) and $(c',m') \in D_n$*

$$(c,m)\ FSD_n\ (c',m') \Rightarrow (c,m) \succeq (c,m)$$

This notion imposes some coherence between preferences $\succeq$ under uncertainty and the preference relation $\succeq_0$ on deterministic consumption path. It is quite a minimalist assumption. It would indeed sound odd to assume that an individual could prefer a lottery which is stochastically dominated at the first order by another one. In the expected utility framework, ordinal dominance is equivalent to assume that the von-Neumann utility index (used to compute expected utility) is increasing with respect to the order that was used for defining first order stochastic dominance. When working in dimension one, with preferences over lotteries with outcome in $\mathbb{R}^+$, and using the natural order of $\mathbb{R}^+$, ordinal dominance involves assuming that the utility index is an increasing function. This is not asking much.

The property of ordinal dominance seems no less desirable when working with temporal lotteries. Though its expression is less obvious. In particular it should be noted that the monotonicity of the aggregator W which has been assumed in Definition 1 is not enough to grant that preferences fulfill ordinal dominance. For example Chew and Epstein (1990) explain that Epstein and Zin isoelastic preferences do not fulfill ordinal dominance, though the aggregator is monotonic. Bommier, Chassagnon and Le Grand (2010) as well as section 6.2 of the current papers highlights some extremely counterintuitive results that can be obtained, when working with preferences which do not fulfill ordinal dominance.

3.2 Representation result

The following result show that imposing recursivity and ordinal dominance readily leaves with a small class of aggregators.

Proposition 1 *A KP-recursive preference relation fulfill ordinal dominance if and only if it can be represented by a KP-recursive utility function that admit one of the following aggregators:*

1.
$$W(x,y) = \alpha(x) + \beta(x)y \tag{7}$$
where $\alpha(\underline{c}) = 0$ *and* $\alpha(\overline{c}) + \beta(\overline{c}) = 1$

2.
$$W(x,y) = \frac{1 - e^{-k(1-\beta)u(x)}(1-\widetilde{k}y)^{\beta}}{\widetilde{k}}$$
where $u(\underline{c}) = 0$, $u(\overline{c}) = 1$, $0 < \beta < 1$ *and* $\widetilde{k} = 1 - e^{-k}$ *for some* $k \neq 0$.

This class is larger than the one of robust preferences, as preferences that are obtained with the aggregator (7) are not of the kind shown in equation (5) when β is not constant. Preferences obtained with (7) are in fact of the expected utility kind and correspond to those that were introduced (in continuous time) by Uzawa (1968) and discussed further in Epstein (1983). These particular cases, where recursivity and ordinal dominance is fulfilled -but preferences are not robust- are however not very useful to discuss risk aversion, as they cannot help to achieve a separation between preferences over deterministic object and risk preferences. The following result end up showing that robust preferences are the only potential candidate to study risk aversion.

Proposition 2 *Consider two KP-recursive preference relations* $\succeq^A$ *and* $\succeq^B$ *on* D. *Assume that their restriction to* D_0 *are identical, denoted by* $\succeq_0$, *and that both* $\succeq^A$ *and* $\succeq^B$ *fulfill the ordinal dominance property with respect to* $\succeq_0$. *Then:*

- *either both preferences are identical:* $\succeq^A = \succeq^B$,
- *or preferences* $\succeq^A$ *and* $\succeq^B$ *can be represented with KP-recursive utility functions, with admissible aggregators* W_A *and* W_B *such that:*

$$W^i = \frac{1 - \exp(-k_i(1-\beta)u(x))(1 - y(1-e^{-k_i}))^{\beta}}{1 - e^{-k_i}} \quad \text{if} \quad k_i \neq 0 \qquad (i = A, B)$$
$$W^i = (1-\beta)u(x) + \beta y \quad \text{if} \quad k_i = 0 \qquad (i = A, B)$$

where u *is a function such that* $u(\underline{c}) = 0$ *and* $u(\overline{c}) = 1$.

The proof of Proposition is relegated in the appendix.

Proposition 2 shows that imposing ordinal preferences and non-trivial comparability of preferences reduces the sets of possible aggregator to those of robust preferences. This therefore indicates that the set of robust preferences, is the only one -within the set of KP-recursive preferences- that may provide an appropriate to study the role of risk aversion. This result was obtained without making precise statements regarding the meaning of comparative risk aversion. We know show that robust preferences are indeed well ordered in terms of risk aversion, even if we use strong notions of comparative risk aversion as that introduced in Bommier, Chassagnon and Le Grand (2010).

4 Comparative risk aversion

As explained in Bommier, Chassagnon and Le Grand (2010) for any riskiness comparison, corresponds a notion of comparative risk aversion.

Definition 3 *(Comparative Risk Aversion). Consider as given a partial order R (-"riskier than"-) defined over the set of temporal lotteries D. Then for any two preference relation $\succeq^A$ and $\succeq^B$ on D, whose restriction $\succeq_0$ on D_0 are identical we will say that the relation of preferences $\succeq^A$ on D is said to exhibit greater (R)-risk aversion than $\succeq^B$ if and only if for all (c,l) and (c',l') we have*

$$\left((c,l)\ R\ (c',l') \text{ and } (c,l)\ \succeq^A\ (c',l')\right) \Rightarrow\ (c,l)\ \succeq^B\ (c',l')$$

Intuitively, if and agent A is more risk averse than B and A considers the increase in risk from (c',l') to (c,l) worthwhile, then agent B should also consider it worthwhile. This procedure to define comparative risk aversion can be attributed to Yaari (1969). It is possible to apply this procedure to different notion of comparative riskiness, that is to different partial orders R. An option consists in focusing, as in Yaari (1969) and most of the subsequent literature, on comparisons between lotteries where at least one is degenerate.

Definition 4 *(Risk Order R_M). For any two (c,l) and $(c',l') \in D$:*

$$(c,l)\ R_M\ (c',l')\ \Leftrightarrow\ (c',l') \in D_0$$

This notion of comparative riskiness is the most minimalist one (from there the denomination R_M). It consists in stating that all degenerate lotteries, with no risk, are minimal elements in terms of the order riskier than. Though it excludes any comparison between two non degenerate lotteries.

Applying Definition 3 to the partial order R_M is by far the most common way in the economics literature to define comparative risk aversion. Kilhstrom and (1974), Chew and Epstein (1990), Epstein and Zin (1989) for example proceed is such a way. It quite trivial to show that:

Proposition 3 *Consider two KP-recursive utility functions U^A and U^B which have the same restriction over D_0. Assume moreover that these KP-recursive utility functions admit two aggregators W^A and W^B given by:*

$$W^i = -\frac{\exp(-k_i(1-\beta)u(x))(1-y(1-e^{-k_i}))^\beta - 1}{1-e^{-k_i}} \quad \text{if} \quad k_i \neq 0 \qquad (i = A, B)$$
$$W^i = (1-\beta)u(x) + \beta y \quad \text{if} \quad k_i = 0 \qquad (i = A, B)$$

with $k_A > k_B$. Then the preferences represented by U^A exhibits more (R_M)-risk aversion than the preferences represented by U^B.

A simple proof could be derived by using the fact that if $k_A > k_B$ then $-\exp(-k_A x)$ is more concave than $-\exp(-k_B x)$. Alternatively that proposition can be seen as a direct consequence of the Proposition 4 derived below.

The above proposition tell us that the class of robust preferences is well ranked in terms of risk aversion, at least when using the less demanding (and also the most common) notion of comparative risk aversion. In Bommier, Chassagnon and Le Grand (2010) it is argued however than focusing on the relation R_M is too minimal since it does not make possible to consider marginal increases or marginal decreases in risk. It is thus suggested to use a stronger notion of comparative riskiness based the single crossing of the cumulative distribution function. The reason to opt for single crossing, rather than for more sophisticated notions of dispersion, like mean preserving spread or second order stochastic dominance, is that it independent from the choice of a particular cardinalisation. This would not be the case when using these other notions of dispersion, as they are not invariant through non-linear monotonic rescaling.

This idea to use single crossing of the cumulative distribution function as an indicator of greater dispersion has however to be adapted to account for the fact that we are working with temporal lotteries, where the notion of first order stochastic dominance had to be defined recursively. This leads to the following definition:

Definition 5 *(Risk Order R_{SC}) For (c,l) and $(c',l') \in D$ we say that (c,l) R_{SC} (c',l') if $(c',l') \in D_0$ or if there exist $n \in \mathbb{N}$ such that (c,l) and $(c',l') \in D_n$ and $p \in [0,1]$ such that*

$$\begin{aligned} (c,l) &= p(c,l_0) + (1-p)(c,l_1) \\ (c,l') &= p(c',l'_0) + (1-p)(c',l_1) \end{aligned}$$

where + denotes the mixture operator and where:

$$(l_0 \times l_1)\{(x_0, x_1) \quad \in \quad D_{n-1} \times D_{n-1} | (c, x_1) FSD_{n-1}(c, x_0)\} = 1$$
$$(l'_0 \times l'_1)\{(x_0, x_1) \quad \in \quad D_{n-1} \times D_{n-1} | (c', x_1) FSD_{n-1}(c', x_0)\} = 1$$

(so that the probability that an outcome of l_1 first order dominates an outcome of l_0 equals 1, and idem for l'_1 and l'_0) and

$$(c', l'_0) \; FSD_n \; (c', l'_0)$$
$$(c, l_1) \; FSD_n \; (c', l'_1)$$

As it defined, it is clear that the notion "riskier than" we introduced is based on comparisons in terms of first-order stochastic dominance, which were itself defined from the relation of preferences $\succeq_0$ over D_0, the set of deterministic consumptions paths. People having different views on the ranking of deterministic consumption paths would therefore have diverging views on whether a lottery is riskier than another one. This could seem restrictive, but before assessing whether a lottery is more or less risky than another one, one needs to be able compare the pay-offs that may be obtained. When pay-off are unidimensional, then comparison is trivial, but when there pay-off are consumption vectors, their comparison may be a matter taste, embedded in the preference relation $\succeq_0$. The important point however, is that the notion of greater riskiness is independent from a specific utility representation for $\succeq_0$. Therefore it does not rely on any aspect that could not be revealed through choices under certainty.

Note also, that just like firs-order stochastic dominance comparison, risk comparison is only possible among lotteries that resolve within a finite amount of time. But, again, we know that $\bigcup_{n \in \mathbb{N}} D_n$ is dense in D, so that restricting risk comparisons to elements of $\bigcup_{n \in \mathbb{N}} D_n$ does not impede to have a meaningful notion of comparative risk aversion.

It is clear that by definition $(c, l) \; R_M \; (c', l') \Rightarrow (c, l) \; R_{SC} \; (c', l')$. Thus the notion of (R_{SC}) comparative risk aversion is stronger than that of (R_M) comparative risk aversion. The following result is therefore stronger than that of Proposition 3:

Proposition 4 *Consider two KP-recursive utility functions U^A and U^B which have the same restriction over D_0. Assume moreover that these KP-recursive utility functions admit two aggregators W^A and W^B given by:*

$$W^i = -\frac{\exp(-k_i(1-\beta)u(x))(1 - y(1 - e^{-k_i}))^{\beta} - 1}{1 - e^{-k_i}} \quad \text{if} \quad k_i \neq 0 \qquad (i = A, B)$$
$$W^i = (1-\beta)u(x) + \beta y \quad \text{if} \quad k_i = 0 \qquad (i = A, B)$$

with $k_A > k_B$. Then the preferences represented by U^A exhibits more (R_{SC})-risk aversion than the preferences represented by U^B.

This proposition establishes that the class of robust preferences, which we obtained in Proposition 2 , is well ordered in terms of risk aversion, even if we use a strong notion of comparative risk aversion based on the R_{SC} risk comparison. We may thus expect that it leads to intuitive results about the role of risk aversion, when applied to problems of asset pricing or risk sharing. Simple problems of the sort are developed in the following section.

5 Applications

In order to show how one can work with robust preferences and get simple conclusions regarding the role of risk aversion, we develop two simple applications: when bears on risk free rate and the risk premium in a random endowment economy. The other on risk sharing in general equilibrium.

5.1 The risk free rate and the risk premium

We consider a (random) endowment economy, in which c_t is random at each date (but typically not $i.i.d$). We focus on the risk free rate and the risk premium. More precisely, we consider the pricing kernel generated by the robust preferences and show that its mean (i.e. the inverse of the risk free rate) and the ratio of its standard deviation to its mean (i.e., the main driver of the risk premium as notably illustrated by the Hansen-Jagannathan (1991) bound) increases with risk aversion.

In the Robust approach, the utility at any date t can be expressed as follows:

$$U_t = (1-\beta)u(c_t) - \frac{\beta}{k}\log(E_t\left[e^{-kU_{t+1}}\right]) \tag{8}$$

We denote $m_{t,t+1}$ as being the pricing kernel associated to previous preferences. The pricing kernel is simply the intertemporal rate of substitution between dates t and $t+1$. We deduce the following expression for the pricing kernel.

$$m_{t,t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}\frac{\exp(-kU_{t+1})}{E[\exp(-kU_{t+1})]} \tag{9}$$

From the pricing kernel we easily deduce the (gross) risk free rate $R_t^{-1} = E[m_{t,t+1}]$ and the market price of risk equal to $\frac{\sigma(m_{t,t+1})}{E[m_{t,t+1}]}$ (where $\sigma(m_{t,t+1})$ denotes the standard deviation of $m_{t,t+1}$). We have the following proposition regarding the impact of risk aversion on risk free rate and market price of risk.

Proposition 5 *Assume that and c_t, U_t, $\frac{e^{kU_t}}{u'(c_t)}$ and $\frac{E[U_{t+1}e^{-kU_{t+1}}]}{E\left[e^{-kU_{t+1}}\right]}$ are comonotonic. Then, a larger risk aversion through a larger k in Eq. (8) defining robust preferences implies:*

- *a smaller risk free rate,*

- *and a larger market price of risk.*

The proof is left in Appendix.

Proposition 5 states that risk aversion in the robust preference framework has a "natural" effect on the risk free rate and the market price of risk. If the agent is more risk averse, he is willing to pay more to transfer resources from a certain state of the world (today) to an uncertain one (tomorrow), which raises the price of riskless savings and thus reduces the riskless interest rate. By the same token, a more risk averse agent requires a larger discount to hold a risky asset, which increases the market price of risk.

As we will see in Section 6.2 the above relationships do not always hold when using Epstein and Zin preferences.

The result derived in Proposition 5 have interesting consequences when discussing very significant policy issues. For example the on-going debate on the cost of climate change heavily bears on the appropriate discount rate to be used, and how this discount rate should be modified to account for risk and the planner's risk aversion. We obtain that the larger is the planner's risk aversion the lower should be the discount rate.

5.2 Risk sharing

In this section we illustrate the behavior of robust preferences in a simple risk-sharing example. We consider an economy populated by two agents denoted A and B. Both agents are endowed with robust preferences. They have the same ordinal preferences but differ with respect to the risk aversion parameter. We assume that agent A is more risk averse than agent B, so that $k_A > k_B$. The utility U_t^i of an agent $i = A, B$ at date t, consuming the amount c_t^i at that date can be expressed as follows

$$U_t^i = (1-\beta)u(c_t^i) - \frac{\beta}{k_i}\log(E_t\left[e^{-k_i U_{t+1}^i}\right]), \tag{10}$$

where β is the constant time discount factor, which the same for both agents. The function u reflects instantaneous utility for consumption and is also identical for both agents.

Both agents are endowed at each date t with the same risky income y_t. We assume that $y_t = y_0 e^{\sigma\varepsilon_t^y}$, where the ε_t^y are iid distributed, with mean 0 and variance σ_y^2. The parameter $\sigma > 0$ is a scale parameter. At each date t, agents have access to two saving technologies. The first one is a riskless asset. Each share costs q_t at date t and pays off one unit of good for sure at the next period $t+1$. The second one is a risky asset, whose price is p_t at date t and which pays off d_t units of good at $t+1$. We assume that $d_t = d_0 e^{\sigma\varepsilon_t^d}$, where the ε_t^d are iid distributed, with mean 0 and variance σ_d^2. The processes ε_t^y and ε_t^d are correlated. Both assets are in aggregate zero net supply and prices adjust to generate a market equilibrium.

The program at date t of an agent $i = A, B$ consists in choosing his consumption stream $\left(c_t^i\right)_{t\geq 0}$, his demands for the riskless asset $\left(b_t^i\right)_{t\geq 0}$ and for the risky asset $\left(a_t^i\right)_{t\geq 0}$, subject to a budget constraint:

$$V^i(a_{t-1}^i, b_{t-1}^i) = \max_{\left(a_t^i, b_t^i, c_t^i\right)_{t\geq 0}} (1-\beta) u\left(c_t^i\right) - \frac{\beta}{k_i} \log\left(E_t\left[e^{-k_i V^i(a_t^i, b_t^i)}\right]\right), \tag{11}$$

$$\text{s.t.} \qquad p_t a_t^i + q_t b_t^i = y_t - c_t^i + d_t a_{t-1}^i + b_{t-1}^i. \tag{12}$$

The budget constraint (12) at date t states that agent's revenue y_t together with payoffs of previous period asset holdings are used either to consume or to purchase risky and riskless assets at respective prices p_t and q_t.

At each date t, both asset markets clear and since they are in zero net supply, we have:

$$a_t^A + a_t^B = 0,$$
$$b_t^A + b_t^B = 0.$$

Writing the Lagrangian of the consumer program (11)–(12), we deduce the following Euler equations for each agent $i = A, B$:

$$\beta E_t\left[\frac{u'(c_{t+1}^i)}{u'(c_t^i)} e^{-k_i V^i(a_t^i, b_t^i)}\right] = q_t E_t\left[e^{-k_i V^i(a_t^i, b_t^i)}\right] \tag{13}$$

$$\beta E_t\left[d_{t+1}\frac{u'(c_{t+1}^i)}{u'(c_t^i)} e^{-k_1 V^i(a_t^i, b_t^i)}\right] = p_t E_t\left[e^{-k_i V^i(a_t^i, b_t^i)}\right] \tag{14}$$

As in Samuelson (1970), we solve the model thanks to an approximation in σ.[3] For any variable x_t, we denote its second order approximation in σ as follows:

$$x_t = x_0 + x_{1,t}\sigma + \frac{1}{2}x_{2,t}\sigma^2 + O(\sigma^3),$$

where x_0 is the constant value of the variable x_t in a certain world. As is derived in appendix we obtain that the asset holding of agent A is given by:

$$a_0^A = -\frac{\frac{k_A - k_B}{2}}{\frac{k_A + k_B}{2} - \frac{u''(c_0)}{(u'(c_0))^2}} \frac{E_t\left[\varepsilon_{t+1}^d \varepsilon_{t+1}^y\right]}{E_t\left[\left(\varepsilon_{t+1}^d\right)^2\right]}. \tag{15}$$

If both shocks commove positively, the more risk averse agent holds less risky asset than the less risk averse one. This is in line with the fact that the asset in that case is a bad hedge against the income risk. We obtain in fact the most the intuitive and expected results : the more risk averse agent takes less risk.

As for the excess return it is found to be equal to:

[3] Another possibility would have been to follow Devereux and Sutherland (2011).

$$\beta E_t\left[\frac{\varepsilon^d_{t+1}\varepsilon^y_{t+1}}{d_0}\right]\left(-\frac{u''(c_0)}{u'(c_0)}+\frac{(1-\beta)u'(c_0)}{2}\left((k_A+k_B)-\frac{\frac{(k_A-k_B)^2}{2}}{\frac{k_A+k_B}{2}-\frac{u''(c_0)}{(u'(c_0))^2}}\right)\right)$$

When both shocks commove positively, the excess return increases with the average risk aversion, but decreases with the heterogeneity in risk aversion (which reflects the demand for risk sharing).

When working with robust preferences, we therefore find that risk aversion has a very simple and intuitive impact.

6 Discussion

In that section we discuss some of aspects of robust preferences, including preferences for the timing. We also explain how these preferences compared with those of Epstein and Zin.

6.1 Preferences for the timing.

As explained in Kreps and Porteus, the concavity of the aggregator that enter in definition 1 with respect to its second argument would be associated with preferences for late resolution of uncertainty, while a convex aggregator would imply preferences for an early resolution of uncertainty. Take

$$1-\widetilde{k}W(x,y)=e^{-k(1-\beta)u(x)}\left[1-\widetilde{k}y\right]^{\beta}\text{ with }\widetilde{k}=1-e^{-k}$$

We have

$$W_y=\beta e^{-k(1-\beta)u(x)}\left[1-\widetilde{k}y\right]^{\beta-1}=\beta\frac{1-\widetilde{k}W(x,y)}{1-\widetilde{k}y}$$

and

$$\frac{W_{yy}}{W_y}=\frac{(1-\beta)\left(1-e^{-k}\right)}{1-\widetilde{k}y}$$

We thus obtain that whenever people are more risk averse than with the standard additive model (thus whenever $k>0$) and whenever people have pure time preferences $\beta<1$, we have a convex aggregator and therefore preferences for an early resolution of uncertainty.

It is noteworthy however, that the less impatient the agent (the closer is β from 1) the weaker are the preferences for the timing. In particular when considering the limit where agents that have no impatience ($\beta=1$) then preferences for the timing would vanish and we would be back to the expected utility framework. That corresponds to the multiplicative model of Bommier (2012), which requires however to restrict the domain of preferences to be defined.

6.2 Comparison with Epstein and Zin preferences

Robust preferences we have been considering share many features with Epstein and Zin isoelastic preferences. They both rely on Kreps and Porteus recursive framework, but use different specifications forms for the aggregators. However this generates significant differences, as to the ability to conform with the ordinal dominance property and to be well ordered in terms of risk aversion.

In order to show why this may end up providing different conclusion we focus on the very simple case of a random endowment economy -which was discussed in Section 5.1 - and investigate whether we could have a different conclusion regarding the risk free rate. To simplify further we assume a very simple form of uncertainty: consumption in time $t+1$ is assumed to be random, but this level of consumption will be maintained for ever. Formally speaking there is a random variable $\widetilde{c}$ such that consumption in all periods after time t is equal to the realization of $\widetilde{c}$. And we look at the risk free rate in period t.

With robust preferences this risk free rate is given by

$$\frac{1}{R_t^{\text{robust}}} = \frac{\beta}{c_t^{\rho-1}} \frac{E\left[\widetilde{c}^{\rho-1}\exp(-k\frac{\widetilde{c}^{\rho}}{\rho})\right]}{E[\exp(-k\frac{\widetilde{c}^{\rho}}{\rho})]}$$

We then have

$$\frac{1}{R_t^{\text{robust}}}\frac{\partial}{\partial k}R_t^{\text{robust}} = \frac{E\left[\frac{\widetilde{c}^{\rho}}{\rho}\widetilde{c}^{\rho-1}\exp(-k\frac{\widetilde{c}^{\rho}}{\rho})\right]}{E\left[\widetilde{c}^{\rho-1}\exp(-k\frac{\widetilde{c}^{\rho}}{\rho})\right]} - \frac{E[\frac{\widetilde{c}^{\rho}}{\rho}\exp(-k\frac{\widetilde{c}^{\rho}}{\rho})]}{E[\exp(-k\frac{\widetilde{c}^{\rho}}{\rho})]}$$

which is negative (whatever the form of the random variable $\widetilde{c}$ and the values of k and ρ) since $\frac{\widetilde{c}^{\rho}}{\rho}$ and $\widetilde{c}^{\rho-1}$ are anticomonotonic.

On the other hand, with Epstein and Zin preferences the risk free rate would be given by:

$$\frac{1}{R_t^{\text{EZ}}} = \frac{\beta}{c_t^{\rho-1}} E\left[\widetilde{c}^{\alpha-1}\right] E\left[\widetilde{c}^{\alpha}\right]^{\frac{\rho}{\alpha}-1}$$

We thus have

$$-\frac{1}{R_t^{\text{EZ}}}\frac{\partial}{\partial\alpha}R_t^{\text{EZ}} = \frac{E\left[\log(c)\widetilde{c}^{\alpha-1}\right]}{E\left[\widetilde{c}^{\alpha-1}\right]} + (\frac{\rho}{\alpha}-1)\frac{E\left[\log(c)\widetilde{c}^{\alpha}\right]}{E\left[\widetilde{c}^{\alpha}\right]} - \frac{\rho}{\alpha^2}\log(E\left[\widetilde{c}^{\alpha}\right])$$

which can be positive or negative. Consider for example the case where $\widetilde{c} = x$ with probability $p << 1$ and $\widetilde{c} = 1$ otherwise. Then:

$$-\frac{1}{R_t^{\text{EZ}}}\frac{\partial}{\partial\alpha}R_t^{\text{EZ}} \simeq p\left[\log(x)x^{\alpha-1} + \left(\frac{\rho}{\alpha}-1\right)\log(x)x^{\alpha} - \frac{\rho}{\alpha^2}(x^{\alpha}-1)\right]$$

Assume that $\alpha > 0$, and $\alpha < \rho$ (people are more risk averse than in the standard additive case). We see that $\frac{\partial}{\partial\alpha}R_t^{\text{EZ}} > 0$ for x close to zero, and $\frac{\partial}{\partial\alpha}R_t^{\text{EZ}} < 0$ for x very large. We would obtain therefore that the risk free rate is non monotonically varying with risk aversion. In the case where we obtain that the risk free rate increase with risk aversion, we would obtain that the willingness to save for precautionary motives decreases with risk aversion, which is contradictory with simple dominance arguments, as was shown in Bommier, Chassagnon and Le Grand (2010).

7 Conclusion

Most of the actions that are undertaken now will have some consequence in the future, which is highly uncertain. It seems therefore crucial for policy guidance to properly account for risk and risk aversion in proper way. Unfortunately their is a dearth of intertemporal economic models that make it possible to discuss the role of risk aversion in satisfying way. This is particularly true when preferences stationarity is imposed, as in most of the macro-economic literature. It occurs indeed that when horizon is possibly infinite, the expected utility framework does not contain any class of stationary preferences which are ordered in terms of risk aversion. The standard approach involves then using Epstein and Zin preferences, despite of serious caveats: they do no fulfill ordinal dominance (Chew and Epstein, 1990) are not well ordered with respect to aversion for marginal increases in risk and leads to strongly counterintuitive conclusions when applied to simple problems (Bommier, Chassagnon and Le Grand (2010)).

In the current paper we explored whether Kreps and Porteus framework could offer better alternatives. We found that if we impose ordinal dominance we are left with a single possibility, which involves using the specification shown in equation (1) and which was introduced -for other purposes - by Hansen and Sargent (1995). We demonstrated that this class of preferences are well ordered in terms of risk aversion, even when using the stronger notion of comparative risk aversion of Bommier, Chassagnon and Le Grand (2010). Illustration using these classes of preferences were developed, showing that they are quite tractable and providing intuitive conclusions with regards the role of risk aversion, contrary to what is sometimes found when working with more common models.

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Appendix

A Proof of Proposition 1

A.1 Necessary conditions

We first prove that if KP-recursive preferences fulfill ordinal dominance, then they can be represented by a KP recursive utility functions admiting on the aggregators given in Proposition 1.

For this purpose we proceed gradually, starting with following Lemma which provide some restrictions on the aggregators.

Lemma 1 *Consider a KP-recursive utility function U defined over D, whose admissible aggregator is denoted W. If the associated preferences function fulfill ordinal dominance, then the aggregator can be expressed as follows:*

$$\forall(x,y) \in [\underline{c},\overline{c}] \times [0,1],\ W(x,y) = \psi\left(\alpha(x) + y\beta(x)\right),$$

where $\alpha:\ [\underline{c},\overline{c}] \to [0,1]$, $\beta:\ [0,1] \to [0,1]$ and $\psi:\ [0,1] \to [0,1]$ are continuously derivable functions.

Proof.

For a given aggregator W one can define preferences over $[\underline{c},\overline{c}] \times M\left([0,1]\right)$ by considering the utility function:

$$\forall(c,m) \in [\underline{c},\overline{c}] \times L\left([0,1]\right),\ V(c,m) = W(c, E[m]),$$

When c varies in $[\underline{c},\overline{c}]$ then $U(c_\infty)$ covers $[0,1]$ and the utility function U -applied to constant consumption path- generates an isomorphism from $[\underline{c},\overline{c}]$ into $[0,1]$. As a consequences if preferences over D fulfill ordinal dominance, preferences over $[\underline{c},\overline{c}] \times M\left([0,1]\right)$ defined as above must also fulfill ordinal dominance. This can be used to find some restrictions on the aggregator W.

First step. The ordinal dominance tells us that for any $(x_0', y_1', y_2') \in [\underline{c},\overline{c}] \times [0,1]^2$, we can find two functions $y_1:\ [\underline{c},\overline{c}]^2 \times [0,1] \to [0,1]$ and $y_2:\ [\underline{c},\overline{c}]^2 \times [0,1] \to [0,1]$, such that the following implication holds for all $x_0 \in [\underline{c},\overline{c}]$ and all $p \in [0,1]$:

$$\left.\begin{array}{r} W(x_0, y_1(x_0, x_0', y_1')) = W(x_0', y_1') \\ W(x_0, y_2(x_0, x_0', y_2')) = W(x_0', y_2') \end{array}\right\} \Rightarrow \qquad (16)$$

$$W(x_0, py_1(x_0,x_0',y_1') + (1-p)y_2(x_0,x_0',y_2')) = W(x_0', py_1' + (1-p)y_2')$$

Since W is continuously differentiable, the implicit function theorem implies that y_1 and y_2 are also continuously differentiable. The derivation of Equation (16) with respect to x_0 then yields:

$$\left.\begin{array}{l} W_x(x_0,y_1)+W_y(x_0,y_1)\frac{\partial y_1}{\partial x_0}(x_0,x_0',y_1')=0 \\ W_x(x_0,y_2)+W_y(x_0,y_2)\frac{\partial y_2}{\partial x_0}(x_0,x_0',y_2')=0 \end{array}\right\} \Rightarrow$$

$$W_x(x_0,py_1+(1-p)y_2)+W_y(x_0,py_1+(1-p)y_2)\left(p\frac{\partial y_1}{\partial x_0}(x_0,y_1',y_2')+(1-p)\frac{\partial y_2}{\partial x_0}(x_0,y_1',y_2')\right)=0, \tag{17}$$

where $\frac{\partial y_i}{\partial x_0}$ $(i=1,2)$ is the partial derivative function of y_i with respect to its first argument, and $W_x(\cdot,\cdot)$ and $W_y(\cdot,\cdot)$ are respectively the partial derivatives of $(x,y)\mapsto W(x,y)$ with respect to the first and the second variable.

After some manipulations, we obtain for all $x_0\in[\underline{c},\overline{c}]$, all $(x_0',y_1',y_2')\in[\underline{c},\overline{c}]\times[0,1]^2$ and all $p\in[0,1]$:

$$\frac{W_x(x_0,py_1(x_0,x_0',y_1')+(1-p)y_2(x_0,x_0',y_2'))}{W_y(x_0,py_1(x_0,x_0',y_1')+(1-p)y_2(x_0,x_0',y_2'))}=p\frac{W_x(x_0,y_1(x_0,x_0',y_1'))}{W_y(x_0,y_1(x_0,x_0',y_1'))}+(1-p)\frac{W_x(x_0,y_2(x_0,x_0',y_2'))}{W_y(x_0,y_2(x_0,x_0',y_2'))}, \tag{18}$$

which implies that $y\mapsto\frac{W_x(x_0,y)}{W_y(x_0,y)}$ is linear on $[0,1]$, for all $x_0\in[\underline{c},\overline{c}]$. Indeed (y_1,y_2) cover $[0,1]^2$ when (x_0',y_1',y_2') cover $[\underline{c},\overline{c}]\times[0,1]^2$.

We deduce that there exist two continuous functions $\widetilde{\alpha}:[\underline{c},\overline{c}]\to[0,1]$ and $\widetilde{\beta}:[\underline{c},\overline{c}]\to[0,1]$,such that $W_x=(\alpha(x)+\beta(x)y)W_y$.

Second step. We need to find a solution to the following linear first order partial differential equation:

$$\begin{aligned}\forall\,(x,y)\in[\underline{c},\overline{c}]\times[0,1],\ W_x(x,y) &= (\widetilde{\alpha}(x)+\widetilde{\beta}(x)y)W_y(x,y)\\ \forall y\in[0,1],\ W(x_0,y) &= \psi(y),\end{aligned}$$

where ψ is an increasing continuously differentiable function.

In that case, the methods of characteristics easily shows the existence and uniqueness of a solution, which is

$$W(x,y)=\psi\left(\int_{x_0}^{x}\left(\int_{x_0}^{a}\widetilde{\beta}(\tau)d\tau\right)\widetilde{\alpha}(a)da+y\,\exp\left(\int_{x_0}^{x}\widetilde{\beta}(a)da\right)\right), \tag{19}$$

which terminates the proof. Indeed it is straightforward that $x\mapsto\int_{x_0}^{x}\left(\int_{x_0}^{a}\widetilde{\beta}(\tau)d\tau\right)\widetilde{\alpha}(a)da$ and $x\mapsto\exp\left(\int_{x_0}^{x}\widetilde{\beta}(a)da\right)$ are continuously derivable on $[0,1]$. $\square$

A.2 A stronger characterization

We now extend generalize the representation result of Lemma 1 by considering streams of consumption, in which the uncertainty is resolved in exactly n periods. We obtain the following lemma which generalizes the result of Lemma 1.

Lemma 2 *We consider a KP-recursive utility function U defined over D, whose admissible aggregator is denoted W. If the utility function fulfills ordinal dominance, then for all $(x_0, x_1, \ldots, x_n) \in [\underline{c}, \overline{c}]^{n+1}$ and for all $y \in [0,1]$, there exist three continuously differentiable functions $\alpha_n : [\underline{c}, \overline{c}]^{n+1} \to [0,1]$, $\beta_n : [\underline{c}, \overline{c}]^{n+1} \to [0,1]$ and $\phi_n : [0,1] \to [0,1]$, with $\phi_n(0) = 0$, $\phi_n(1) = 1$, $\alpha_n(\underline{c}, \underline{c}, .., \underline{c}) = 0$ and $\alpha_n(\overline{c}, \overline{c}, ..., \overline{c}) + \beta_n(\overline{c}, \overline{c}, ..., \overline{c}) = 1$, such that the aggregator W can be expressed as follows:*

$$\forall (x_0, x_1, \ldots, x_n, y) \in [\underline{c}, \overline{c}]^{n+1} \times [0,1],$$
$$W(x_0, W(x_1, W(x_2, \ldots, W(x_n, y)))) = \phi_n \left(\alpha_n \left(x_0, x_1, ..., x_n\right) + \beta_n \left(x_0, x_1, ..., x_n\right) y\right). \tag{20}$$

Proof. We prove the result by induction.

1. For $n = 0$, the result directly stems from Lemma 1.

2. We assume that the result holds for a given $n \in \mathbb{N}$. We consider $(x_0, x_1, \ldots, x_{n+1}) \in [\underline{c}, \overline{c}]^{n+2}$ and $y \in [0,1]$. Using the induction hypothesis, we know that there exist three continuously derivable functions $\alpha_n : [\underline{c}, \overline{c}]^{n+1} \to [0,1]$, $\beta_n : [0,1]^{n+1} \to [0,1]$ and $\phi_n : [0,1] \to [0,1]$, such that

$$W(x_1, W(x_1, W(x_2, \ldots, W(x_{n+1}, y)))) = \phi_n \left(\alpha_n \left(x_1, ..., x_{n+1}\right) + \beta_n \left(x_1, ..., x_{n+1y}\right) y\right)$$

From Lemma 1, we also know that there exist three continuously derivable functions $\alpha : [\underline{c}, \overline{c}] \to [0,1]$, $\beta : [0,1] \to [0,1]$ and $\psi : [0,1] \to [0,1]$, such that $\forall (x,y) \in [\underline{c}, \overline{c}] \times [0,1]$, $W(x,y) = \psi\left(\alpha(x) + y\beta(x)\right)$. We therefore deduce the following equality:

$$W(x_0, W(x_1, W(x_2, \ldots, W(x_{n+1}, y)))) = \psi\left(\alpha(x_0) + \beta(x_0)\phi_n\left(\alpha_n\left(x_1, ..., x_{n+1}\right) + \beta_n\left(x_1, ..., x_{n+1}\right) y\right)\right) \tag{21}$$

Finally we use a similar argument as in the Proof of Lemma 1. As in (16), we consider $(x_1, \ldots, x_n) \in [\underline{c}, \overline{c}]^n$ and$(x_0', y_1', y_2') \in [\underline{c}, \overline{c}] \times [0,1]^2$. We can find two functions $y_1 : [\underline{c}, \overline{c}]^{n+2} \times [0,1] \to [0,1]$ and $y_2 : [\underline{c}, \overline{c}]^{n+2} \times [0,1] \to [0,1]$, such that the following implication holds for all $x_0 \in [\underline{c}, \overline{c}]$ and all $p \in [0,1]$ (we have skipped arguments to clarify notations):

$$\left.\begin{array}{r}\psi\left(\alpha(x_0) + \beta(x_0)\phi_n\left(\alpha_n + \beta_n y_1\right)\right) = \psi\left(\alpha(x_0') + \beta(x_0')\phi_n\left(\alpha_n + \beta_n y_1'\right)\right) \\ \psi\left(\alpha(x_0) + \beta(x_0)\phi_n\left(\alpha_n + \beta_n y_2\right)\right) = \psi\left(\alpha(x_0') + \beta(x_0')\phi_n\left(\alpha_n + \beta_n y_2'\right)\right)\end{array}\right\} \Rightarrow$$
$$\psi\left(\alpha(x_0) + \beta(x_0)\phi_n\left(\alpha_n + \beta_n\left(py_1 + (1-p)y_2\right)\right)\right) = \psi\left(\alpha(x_0') + \beta(x_0')\phi_n\left(\alpha_n + \beta_n\left(py_1' + (1-p)y_2'\right)\right)\right)$$

We derive with respect to $x_0$and obtain a set of equations similar to (17) (arguments have been skipped):

$$\left.\begin{array}{l}\big(\alpha'(x_0)+\beta'(x_0)\phi_n\left(\alpha_n+\beta_n y_1\right)\big)+\beta(x_0)\beta_n\phi_n'\left(\alpha_n+\beta_n y_1\right)\frac{\partial y_1}{\partial x_0}=0\\ \big(\alpha'(x_0)+\beta'(x_0)\phi_n\left(\alpha_n+\beta_n y_2\right)\big)+\beta(x_0)\beta_n\phi_n'\left(\alpha_n+\beta_n y_2\right)\frac{\partial y_2}{\partial x_0}=0\end{array}\right\}\Rightarrow$$

$$\begin{aligned}&\big(\alpha'(x_0)+\beta'(x_0)\phi_n\left(\alpha_n+\beta_n\left(py_1+(1-p)y_2\right)\right)\big)\\&+\beta(x_0)\beta_n\phi_n'\left(\alpha_n+\beta_n\left(py_1+(1-p)y_2\right)\right)\left(p\frac{\partial y_1}{\partial x_0}+(1-p)\frac{\partial y_2}{\partial x_0}\right)=0,\end{aligned}$$

which simplifies after some manipulations to an equality similar to (18):

$$\begin{aligned}&\frac{\alpha'(x_0)+\beta'(x_0)\phi_n\left(\alpha_n+\beta_n\left(py_1+(1-p)y_2\right)\right)}{\beta(x_0)\beta_n\phi_n'\left(\alpha_n+\beta_n\left(py_1+(1-p)y_2\right)\right)}=\\ p&\frac{\alpha'(x_0)+\beta'(x_0)\phi_n\left(\alpha_n+\beta_n y_1\right)}{\beta(x_0)\beta_n\phi_n'\left(\alpha_n+\beta_n y_1\right)}+(1-p)\frac{\alpha'(x_0)+\beta'(x_0)\phi_n\left(\alpha_n+\beta_n y_2\right)}{\beta(x_0)\beta_n\phi_n'\left(\alpha_n+\beta_n y_2\right)}\end{aligned}$$

We deduce that $y\mapsto\frac{\alpha'(x_0)+\beta'(x_0)\phi_n(\alpha_n+\beta_n y)}{\beta(x_0)\beta_n\phi_n'(\alpha_n+\beta_n y)}$ is linear. Moreover, we observe that $\frac{\alpha'(x_0)+\beta'(x_0)\phi_n(\alpha_n+\beta_n y)}{\beta(x_0)\beta_n\phi_n'(\alpha_n+\beta_n y)}=\frac{\frac{\partial}{\partial x_0}\alpha(x_0)+\beta(x_0)\phi_n(\alpha_n+\beta_n y)}{\frac{\partial}{\partial y}\alpha(x_0)+\beta(x_0)\phi_n(\alpha_n+\beta_n y)}$. As in (19), the method of characteristics allows us to find three continuously differentiable functions $\alpha_{n+1}:[\underline{c},\overline{c}]^{n+2}\to[0,1]$, $\beta_{n+1}:[\underline{c},\overline{c}]^{n+2}\to[0,1]$ and $\psi_{n+1}:[0,1]\to[0,1]$, such that:

$$\alpha(x_0)+\beta(x_0)\phi_n\left(\alpha_n\left(x_1,...,x_{n+1}\right)+\beta_n\left(x_1,...,x_{n+1}\right)y\right)=\psi_{n+1}\left(\alpha_{n+1}\left(x_0,...,x_{n+1}\right)+\beta_{n+1}\left(x_0,...,x_{n+1}\right)y\right).$$

We deduce from (21) that:

$$W(x_0,W(x_1,\dots,W(x_{n+1},y)))=\psi\circ\psi_{n+1}\left(\alpha_{n+1}\left(x_0,...,x_{n+1}\right)+\beta_n\left(x_0,...,x_{n+1}\right)y\right).$$

Denoting $\phi_{n+1}=\psi\circ\psi_{n+1}:[0,1]\to[0,1]$, which is continuously differentiable, concludes the proof.

■

The following lemma characterizes the functional form of the aggregator when preferences fulfill ordinal dominance. More precisely, we derive functional restrictions imposed when Equation (20) of Lemma 2 holds.

Lemma 3 *We consider a KP-recursive utility function U defined over D, whose admissible aggregator is denoted W. We assume that the function W is such that for any $n\in\mathbb{N}$, there exist three continuously derivable functions $\alpha_n:[\underline{c},\overline{c}]^{n+1}\to[0,1]$, $\beta_n:[0,1]^{n+1}\to[0,1]$ and $\phi_n:[0,1]\to[0,1]$, such that*

$$\begin{aligned}&\forall\left(x_0,x_1,\dots,x_n,y\right)\in[\underline{c},\overline{c}]^{n+1}\times[0,1],\\&W(x_0,W(x_1,W(x_2,\dots,W(x_n,y))))=\phi_n\left(\alpha_n\left(x_0,x_1,...,x_n\right)+\beta_n\left(x_0,x_1,...,x_n\right)y\right).\end{aligned}$$

Then, there exist three continuously derivable functions $\alpha_0 : [\underline{c}, \overline{c}] \to [0,1]$, $\beta_0 : [0,1] \to [0,1]$ *and* $\phi_0 : [0,1] \to [0,1]$, *such that:*

$$\forall (x,y) \in [\underline{c}, \overline{c}] \times [0,1], \quad W(x,y) = \phi_0(\alpha_0(x) + y\beta_0(x))$$

where α_0, β_0 *and* ϕ_0 *verify one of the following properties:*

1.
$$W(x,y) = \alpha(x) + \beta(x)y \tag{22}$$
where $\alpha(\underline{c}) = 0$ *and* $\alpha(\overline{c}) + \beta(\overline{c}) = 1$

2.
$$W(x,y) = \frac{1 - e^{-k(1-\beta)u(x)}(1-\widetilde{k}y)^{\beta}}{\widetilde{k}}$$

Proposition 6 *1. where* $u(\underline{c}) = 0$, $u(\overline{c}) = 1$, $0 < \beta < 1$ *and* $\widetilde{k} = 1 - e^{-k}$ *for some* $k \neq 0$.

Proof.

We proceed in 3 steps to derive necessary conditions implied by (20).

1. We assume that (20) holds for $n = 0$.

$$\forall (x_0, y) \in [\underline{c}, \overline{c}] \times [0,1], \phi_0^{-1} \circ W(x_0, y) = \alpha_0(x_0) + \beta_0(x_0)\, y.$$

The ratio of the partial derivatives wrt x_0 and y provides the following equality for all $(x,y) \in [\underline{c}, \overline{c}] \times [0,1]$:

$$\frac{W_x(x,y)}{W_y(x,y)} = \frac{\alpha_0'(x)}{\beta_0(x)} + \frac{\beta_0'(x)}{\beta_0(x)} y. \tag{23}$$

2. We assume that (20) holds for $n = 1$.

$$\forall (x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1], \phi_1^{-1} \circ W(x_0, W(x_1, y)) = \alpha_1(x_0, x_1) + \beta_1(x_0, x_1)\, y.$$

The ratios of the partial derivatives wrt x_0 and y on one hand and x_1 and y on the other hand provide the following equalities:

$$\frac{W_x(x_0, W(x_1,y))}{W_y(x_1,y)W_y(x_0, W(x_1,y))} = \frac{\alpha_{1,x_0}(x_0,x_1)}{\beta_1(x_0,x_1)} + \frac{\beta_{1,x_0}(x_0,x_1)}{\beta_1(x_0,x_1)} y, \tag{24}$$

$$\frac{W_x(x_1,y)W_y(x_0, W(x_1,y))}{W_y(x_1,y)W_y(x_0, W(x_1,y))} = \frac{W_x(x_1,y)}{W_y(x_1,y)} = \frac{\alpha_{1,x_1}(x_0,x_1)}{\beta_1(x_0,x_1)} + \frac{\beta_{1,x_1}(x_0,x_1)}{\beta_1(x_0,x_1)} y \tag{25}$$

We deduce from (23) and (25) the following equality, which holds for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{\alpha_{1,x_1}(x_0,x_1)}{\beta_1(x_0,x_1)} = \frac{\alpha_0'(x_1)}{\beta_0(x_1)},$$
$$\frac{\beta_{1,x_1}(x_0,x_1)}{\beta_1(x_0,x_1)} = \frac{\beta_0'(x_1)}{\beta_0(x_1)},$$

or after integration there exist two functions $a_0,\ b_0 : [\underline{c}, \overline{c}] \to [0,1]$, such that:

$$\beta_1(x_0, x_1) = b_0(x_0)\beta_0(x_1), \tag{26}$$

$$\alpha_1(x_0, x_1) = b_0(x_0)\alpha_0(x_1) + a_0(x_0). \tag{27}$$

Substituting (26) and (27) into (23) and (24) implies that for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$, we have:

$$\frac{\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y)}{W_y(x_1, y)} = \frac{b_0'(x_0)\alpha_0(x_1) + a_0'(x_0)}{b_0(x_0)\beta_0(x_1)} + \frac{b_0'(x_0)}{b_0(x_0)} y. \tag{28}$$

Deriving (28) wrt to y and rearranging the equality yields for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y)}{W_y(x_1, y)} \frac{W_{yy}(x_1, y)}{W_y(x_1, y)} = \frac{\beta_0'(x_0)}{\beta_0(x_0)} - \frac{b_0'(x_0)}{b_0(x_0)}. \tag{29}$$

It implies that $\frac{\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y)}{W_y(x_1, y)} \frac{W_{yy}(x_1, y)}{W_y(x_1, y)}$ does not depend on y and x_1.

The first possibility is to have $\frac{\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y)}{W_y(x_1, y)} \frac{W_{yy}(x_1, y)}{W_y(x_1, y)} = 0$, for all y and x_1. But from (23) we know that $\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} y > 0$ for all $y \in [0,1]$. Thus $\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y) > 0$ and it must be the case $W_{yy}(x_1, y) = 0$ for all y and x_1, which leads to the linear aggregator (7).

Otherwise if $W_{yy} \neq 0$ we may log-derive $\frac{\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y)}{W_y(x_1, y)} \frac{W_{yy}(x_1, y)}{W_y(x_1, y)}$ wrt y, which gives that for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{\partial}{\partial y} \ln\left(\frac{\alpha_0'(x_0)}{\beta_0(x_0)} + \frac{\beta_0'(x_0)}{\beta_0(x_0)} W(x_1, y)\right) + \frac{\partial}{\partial y} \ln\left(\frac{W_{yy}(x_1, y)}{W_y(x_1, y)^2}\right) = 0,$$

$$\frac{\beta_0'(x_0) W_y(x_1, y)}{\alpha_0'(x_0) + \beta_0'(x_0) W(x_1, y)} + \frac{\partial}{\partial y} \ln\left(\frac{W_{yy}(x_1, y)}{W_y(x_1, y)^2}\right) = 0.$$

We deduce that there exists $\lambda_0 \in \mathbb{R}$, such that for all $x_0 \in [\underline{c}, \overline{c}]$,

$$\beta_0'(x_0) = \lambda_0 \alpha_0'(x_0). \tag{30}$$

We first assume that there exists x_0, such that $\beta_0'(x_0) \neq 0$, which implies $\lambda_0 \neq 0$ (the case $\lambda_0 = 0$ will be treated later on). Equation (28) becomes:

$$\frac{\alpha_0'(x_0)}{\beta_0(x_0)} \frac{1 + \lambda_0 W(x_1, y)}{W_y(x_1, y)} = \frac{b_0'(x_0)\alpha_0(x_1) + a_0'(x_0)}{b_0(x_0)\beta_0(x_1)} + \frac{b_0'(x_0)}{b_0(x_0)} y. \tag{31}$$

Substituting (30) into (29) provides the following equality holding for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{\alpha_0'(x_0)}{\beta_0(x_0)} \left(\lambda_0 - \frac{1 + \lambda_0 W(x_1, y)}{W_y(x_1, y)} \frac{W_{yy}(x_1, y)}{W_y(x_1, y)}\right) = \frac{b_0'(x_0)}{b_0(x_0)}.$$

We deduce that there exists $\mu_0 \in \mathbb{R}$, such that for all $x_0 \in [\underline{c}, \overline{c}]$,

$$\frac{b_0'(x_0)}{b_0(x_0)} = \mu_0 \frac{\alpha_0'(x_0)}{\beta_0(x_0)} \left(= \frac{\mu_0}{\lambda_0} \frac{\beta_0'(x_0)}{\beta_0(x_0)}\right). \tag{32}$$

We can notice that $\mu_0 = 0$ implies $\beta_0'(x_0) = 0$ for all x_0, that we have ruled out. Equation (31) becomes after plugging (32) (for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$):

$$\frac{\alpha_0'(x_0)}{\beta_0(x_0)} \left(\frac{1 + \lambda_0 W(x_1, y)}{W_y(x_1, y)} - \mu_0 y - \mu_0 \frac{\alpha_0(x_1)}{\beta_0(x_1)} \right) = \frac{a_0'(x_0)}{b_0(x_0)\beta_0(x_1)}. \tag{33}$$

We deduce that there exists $\nu_0 \in \mathbb{R}$, such that for all $x_0 \in [\underline{c}, \overline{c}]$,

$$\begin{aligned} \frac{a_0'(x_0)}{b_0(x_0)} &= \nu_0 \frac{\alpha_0'(x_0)}{\beta_0(x_0)}, \\ a_0'(x_0) &= \frac{\nu_0}{\mu_0} b_0'(x_0). \end{aligned} \tag{34}$$

We obtain after substitution of (34) into (33) for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{W_y(x_1, y)}{1 + \lambda_0 W(x_1, y)} = \frac{1}{\frac{\mu_0 \alpha_0(x_1) + \nu_0}{\beta_0(x_1)} + \mu_0 y}.$$

We deduce that there exists $\kappa : [\underline{c}, \overline{c}] \to [0,1]$, such that for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\mu_0 \ln(1 + \lambda_0 W(x_1, y)) = \lambda_0 \ln(\kappa(x_1)) + \lambda_0 \ln\left(\frac{\mu_0 \alpha_0(x_1) + \nu_0}{\beta_0(x_1)} + \mu_0 y \right) \tag{35}$$

We derive (35) wrt x_1 and y and we obtain for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\begin{aligned} \frac{\lambda_0 \mu_0 W_x(x_1, y)}{1 + \lambda_0 W(x_1, y)} &= \lambda_0 \frac{\kappa'(x_1)}{\kappa(x_1)} + \frac{\lambda_0 \frac{\alpha_0'(x_1)}{\beta_0(x_1)} \left(\mu_0 - \frac{\mu_0 \alpha_0(x_1) + \nu_0}{\beta_0(x_1)} \lambda_0 \right)}{\frac{\mu_0 \alpha_0(x_1) + \nu_0}{\beta_0(x_1)} + \mu_0 y}, \\ \frac{\lambda_0 \mu_0 W_y(x_1, y)}{1 + \lambda_0 W(x_1, y)} &= \frac{\lambda_0 \mu_0}{\frac{\mu_0 \alpha_0(x_1) + \nu_0}{\beta_0(x_1)} + \mu_0 y}. \end{aligned}$$

The ratio of both previous equation yields for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{W_x(x_1, y)}{W_y(x_1, y)} = \frac{\alpha_0'(x_1)}{\beta_0(x_1)} \left(1 - \frac{\alpha_0(x_1) + \frac{\nu_0}{\mu_0}}{\beta_0(x_1)} \lambda_0 \right) + \left(\frac{\alpha_0(x_1) + \frac{\nu_0}{\mu_0}}{\beta_0(x_1)} + y \right) \frac{\kappa'(x_1)}{\kappa(x_1)}, \tag{36}$$

while we already know from (23) together with (30) that for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{W_x(x_1, y)}{W_y(x_1, y)} = \frac{\alpha_0'(x_1)}{\beta_0(x_1)} (1 + \lambda_0 y). \tag{37}$$

Comparing (36) with (37), we obtain the following equality for all $x_1 \in [\underline{c}, \overline{c}]$:

$$\frac{\kappa'(x_1)}{\kappa(x_1)} = \lambda_0 \frac{\alpha_0'(x_1)}{\beta_0(x_1)} = \frac{\beta_0'(x_1)}{\beta_0(x_1)}. \tag{38}$$

We deduce from (30), (32) (34) and (38) that there exist $(\kappa_0, b, a, \alpha) \in \mathbb{R}^4$, such that for all $(x_0, x_1) \in [\underline{c}, \overline{c}]^2$:

$$\kappa(x_1) = \kappa_0 \beta_0(x_1), \tag{39}$$

$$b(x_0) = b \beta_0(x_0)^{\frac{\mu_0}{\lambda_0}}, \tag{40}$$

$$a_0(x_0) = \frac{\nu_0}{\mu_0} \left(b \beta_0(x_0)^{\frac{\mu_0}{\lambda_0}} + a \right), \tag{41}$$

$$\beta_0(x_1) = \lambda_0 (\alpha_0(x_1) + \alpha). \tag{42}$$

By substitution into (35), we obtain for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$\begin{aligned}
\mu_0 \ln(1 + \lambda_0 W(x_1, y)) &= \lambda_0 \ln(\kappa_0) + \lambda_0 \ln(\mu_0 \alpha_0(x_1) + \nu_0 + \lambda_0 \mu_0 (\alpha_0(x_1) + \alpha) y) \\
1 + \lambda_0 W(x_1, y) &= \kappa_0^{\frac{\lambda_0}{\mu_0}} (\mu_0 \alpha_0(x_1) + \nu_0 + \lambda_0 \mu_0 (\alpha_0(x_1) + \alpha) y)^{\frac{\lambda_0}{\mu_0}} \\
&= (\kappa_0 \nu_0 + \kappa_0 \mu_0 (\alpha_0(x_1) + \beta_0(x_1) y))^{\frac{\lambda_0}{\mu_0}} \qquad (43)
\end{aligned}$$

It is useful to observe that deriving (43) wrt x and y yields the following equalities holding for all $(x, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$\begin{aligned}
W_x(x, y) &= \kappa_0 \beta_0'(x) \left(\lambda_0^{-1} + y\right) (\kappa_0 \nu_0 + \kappa_0 \mu_0 (\alpha_0(x) + \beta_0(x) y))^{\frac{\lambda_0}{\mu_0} - 1} \qquad (44) \\
W_y(x, y) &= \kappa_0 \beta_0(x) (\kappa_0 \nu_0 + \kappa_0 \mu_0 (\alpha_0(x) + \beta_0(x) y))^{\frac{\lambda_0}{\mu_0} - 1} \qquad (45) \\
\frac{W_x(x, y)}{W_y(x, y)} &= \lambda_0^{-1} \frac{\beta_0'(x)}{\beta_0(x)} (1 + \lambda_0 y) \qquad (46) \\
\frac{1 + \lambda_0 W(x, y)}{W_y(x, y)} &= \frac{\lambda_0}{\mu_0} \frac{\nu_0 + \mu_0 (\alpha_0(x) + \beta_0(x) y)}{\beta_0(x)} \qquad (47)
\end{aligned}$$

3. We assume that (20) holds for $n = 2$.

$$\forall (x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1], \phi_2^{-1} \circ W(x_0, W(x_1, W(x_2, y))) = \alpha_2 (x_0, x_1, x_2) + \beta_2 (x_0, x_1, x_2) y.$$

The ratio of the partial derivatives wrt x_0 and y provides the following equality holding for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$:

$$\begin{aligned}
J(x_0, x_1, x_2, y) &= \frac{W_x(x_0, W(x_1, W(x_2, y)))}{W_y(x_2, y) W_y(x_1, W(x_2, y)) W_y(x_0, W(x_1, W(x_2, y)))}, \\
&= \frac{\alpha_{2,x_0}(x_0, x_1, x_2)}{\beta_2(x_0, x_1, x_2)} + \frac{\beta_{1,x_0}(x_0, x_1, x_2)}{\beta_2(x_0, x_1, x_2)} y. \qquad (48)
\end{aligned}$$

We want to simplify the expression $J(x_0, x_1, x_2, y)$. First, using Equation (46), we have for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$:

$$J(x_0, x_1, x_2, y) = \lambda_0^{-1} \frac{\beta_0'(x_0)}{\beta_0(x_0)} \frac{1}{W_y(x_2, y)} \frac{1 + \lambda_0 W(x_1, W(x_2, y))}{W_y(x_1, W(x_2, y))}.$$

From Equation (47), we obtain for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$:

$$J(x_0, x_1, x_2, y) = \frac{1}{\mu_0} \frac{\beta_0'(x_0)}{\beta_0(x_0) \beta_0(x_1)} \frac{\nu_0 + \mu_0 (\alpha_0(x_1) + \beta_0(x_1) W_y(x_2, y))}{W_y(x_2, y)}.$$

Plugging the expression (42) of $\beta_0(\cdot)$, we deduce that for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$, we have:

$$\begin{aligned}
\left(\frac{1}{\mu_0} \frac{\beta_0'(x_0)}{\beta_0(x_0) \beta_0(x_1)}\right)^{-1} J(x_0, x_1, x_2, y) &= \frac{\nu_0 + \mu_0 (\alpha_0(x_1) + \lambda_0 (\alpha_0(x_1) + \alpha) W(x_2, y))}{W_y(x_2, y)}, \\
&= \frac{\nu_0 + \mu_0 \lambda_0 \alpha W(x_2, y)}{W_y(x_2, y)} + \alpha_0(x_1) \mu_0 \frac{1 + \lambda_0 W(x_2, y)}{W_y(x_2, y)} \\
&= \frac{\nu_0 - \mu_0 \alpha}{W_y(x_2, y)} + (\alpha_0(x_1) + \alpha) \mu_0 \frac{1 + \lambda_0 W(x_2, y)}{W_y(x_2, y)}. \qquad (49)
\end{aligned}$$

Equation (48) states that (49) should be an affine function of $y \in [0,1]$. Using (45) and (47), we deduce that it implies $\alpha = \frac{\nu_0}{\mu_0}$ (since $\lambda_0 \neq 0$). We deduce that (42) becomes for all $x \in [\underline{c}, \overline{c}]^2$:

$$\beta_0(x_1) = \frac{\lambda_0}{\mu_0} \left(\mu_0 \alpha_0(x_1) + \nu_0\right). \tag{50}$$

Using the above expression (50) of $\beta_0(\cdot)$, we can simplify the expression (43) of the aggregator W, which becomes for all $(x,y) \in [\underline{c}, \overline{c}] \times [0,1]$:

$$\begin{aligned} 1 + \lambda_0 W(x,y) &= \left(\kappa_0 \nu_0 + \kappa_0 \mu_0 \left(\alpha_0(x) + \beta_0(x) y\right)\right)^{\frac{\lambda_0}{\mu_0}}, \\ &= \left(\kappa_0 \left(\nu_0 + \mu_0 \alpha_0(x)\right)\left(1 + \lambda_0 y\right)\right)^{\frac{\lambda_0}{\mu_0}}. \end{aligned} \tag{51}$$

This gives:

$$W(x,y) = \frac{\left(\kappa_0 \left(\nu_0 + \mu_0 \alpha_0(x)\right)\left(1 + \lambda_0 y\right)\right)^{\frac{\lambda_0}{\mu_0}} - 1}{\lambda_0}$$

Now by posing $\lambda_0 = -\widetilde{k}$, $k = -\log(1+\lambda_0)$, $\beta = \frac{\lambda_0}{\mu_0}$ and $u(x) = \frac{-\beta}{k(1-\beta)} \log\left(\kappa_0 \left(\nu_0 + \mu_0 \alpha_0(x)\right)\right)$ we obtain:

$$\begin{aligned} W(x,y) &= \frac{1 - \left(\kappa_0 \left(\nu_0 + \mu_0 \alpha_0(x)\right)\left(1 - \widetilde{k} y\right)\right)^{\beta}}{\widetilde{k}} \\ &= \frac{1 - \left(\kappa_0 \left(\nu_0 + \mu_0 \alpha_0(x)\right)\right)^{\beta}\left(1 - \widetilde{k} y\right)^{\beta}}{\widetilde{k}} \\ &= \frac{1 - e^{-k(1-\beta)u(x)}(1 - \widetilde{k} y)^{\beta}}{\widetilde{k}} \end{aligned}$$

which provides therefore specification.

4. We now consider the case where we would have $W_{yy} \neq 0$ and $\lambda_0 = 0$, and show that this is impossible. From $\lambda_0 = 0$ we deduce that for all $x \in [\underline{c}, \overline{c}]$, $\beta_0(x) = \beta_0$. Equations (26) and (27) become for all $(x_0, x_1) \in [\underline{c}, \overline{c}]^2$:

$$\beta_1(x_0, x_1) = b_0(x_0)\beta_0, \tag{52}$$

$$\alpha_1(x_0, x_1) = b_0(x_0)\alpha_0(x_1) + a_0(x_0). \tag{53}$$

Substituting (52) and (53) into (23) and (24) implies that for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$, we have:

$$\frac{\alpha_0'(x_0)}{\beta_0} \frac{W(x_1,y)}{W_y(x_1,y)} = \frac{b_0'(x_0)\alpha_0(x_1) + a_0'(x_0)}{b_0(x_0)\beta_0} + \frac{b_0'(x_0)}{b_0(x_0)} y. \tag{54}$$

The above equation implies that there exists $\mu_0 \neq 0$, such that $\frac{b_0'(x_0)}{b_0(x_0)} = \mu_0 \frac{\alpha_0'(x_0)}{\beta_0}$ (indeed, if $\mu_0 = 0$, we have W_y constant, which is a case that we rule out). Equation (54) becomes then for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0,1]$:

$$\frac{\alpha_0'(x_0)}{\beta_0} \left(\frac{W(x_1,y)}{W_y(x_1,y)} - \mu_0 y - \mu_0 \frac{\alpha_0(x_1)}{\beta_0} \right) = \frac{a_0'(x_0)}{b_0(x_0)\beta_0}. \tag{55}$$

We deduce that there exists $\nu_0 \in \mathbb{R}$, such that for all $x_0 \in [\underline{c}, \overline{c}]$,

$$\frac{a_0'(x_0)}{b_0(x_0)} = \nu_0 \alpha_0'(x_0).$$

After substitution in (55), we obtain for all $(x_0, x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$

$$\frac{W_y(x_1, y)}{W(x_1, y)} = \frac{1}{\mu_0 y + \mu_0 \frac{\alpha_0(x_1)}{\beta_0} + \nu_0}.$$

We deduce that there exists $\kappa : [\underline{c}, \overline{c}] \to [0, 1]$, such that for all $(x_1, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$W(x_1, y) = \kappa(x_1) \left(\frac{\mu_0 \alpha_0(x_1)}{\beta_0} + \nu_0 + \mu_0 y \right)^{\frac{1}{\mu_0}}. \tag{56}$$

We log-derive (56) wrt x_1 and y and we obtain for all $(x, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$\begin{aligned}
\frac{W_x(x, y)}{W(x, y)} &= \frac{\kappa'(x)}{\kappa(x)} + \frac{1}{\beta_0} \frac{\alpha_0'(x)}{\frac{\mu_0 \alpha_0(x)}{\beta_0} + \nu_0 + \mu_0 y}, \\
\frac{W_y(x_1, y)}{W(x_1, y)} &= \frac{1}{\mu_0 y + \mu_0 \frac{\alpha_0(x)}{\beta_0} + \nu_0}.
\end{aligned}$$

The ratio of both previous equation yields for all $(x, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$\frac{W_x(x, y)}{W_y(x, y)} = \frac{\alpha_0'(x)}{\beta_0} + \frac{\kappa'(x)}{\kappa(x)} \left(\frac{\mu_0 \alpha_0(x)}{\beta_0} + \nu_0 + \mu_0 y \right), \tag{57}$$

while we already know from (23) together with (30) that for all $(x, y) \in [\underline{c}, \overline{c}] \times [0, 1]$:

$$\frac{W_x(x, y)}{W_y(x, y)} = \frac{\alpha_0'(x)}{\beta_0}. \tag{58}$$

Comparing (57) with (58), we obtain the following equality for all $(x, y) \in [\underline{c}, \overline{c}] \times [0, 1]$:

$$\kappa'(x) \left(\frac{\mu_0 \alpha_0(x)}{\beta_0} + \nu_0 + \mu_0 y \right) = 0,$$

which implies

$$\kappa'(x) = 0. \tag{59}$$

We deduce from (30), (32) (34) and (59) that there exist $(\kappa_0, b, a, \alpha) \in \mathbb{R}^4$, such that for all $x \in [\underline{c}, \overline{c}]$:

$$\kappa(x) = \kappa_0, \tag{60}$$

$$b(x) = b e^{\frac{\mu_0}{\beta_0} \alpha_0(x)}, \tag{61}$$

$$a_0(x_0) = \frac{\nu_0}{\mu_0} \beta_0 \left(b e^{\frac{\mu_0}{\beta_0} \alpha_0(x)} + a \right), \tag{62}$$

$$\beta_0(x) = \beta_0. \tag{63}$$

By substitution into (56), we obtain for all $(x, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$W(x, y) = \kappa_0 \left(\frac{\mu_0 \alpha_0(x)}{\beta_0} + \nu_0 + \mu_0 y \right)^{\frac{1}{\mu_0}}. \tag{64}$$

We can remark that for all $(x, y) \in [\underline{c}, \overline{c}]^2 \times [0, 1]$:

$$W_y(x, y) = \kappa_0 \left(\frac{\mu_0 \alpha_0(x)}{\beta_0} + \nu_0 + \mu_0 y \right)^{\frac{1}{\mu_0} - 1}. \tag{65}$$

We assume that (20) holds for $n = 2$.

$\forall (x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1], \phi_2^{-1} \circ W(x_0, W(x_1, W(x_2, y))) = \alpha_2 (x_0, x_1, x_2) + \beta_2 (x_0, x_1, x_2)\, y.$

The ratio of the partial derivatives wrt x_0 and y provides the following equality holding for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$:

$$\begin{aligned} J(x_0, x_1, x_2, y) &= \frac{W_x(x_0, W(x_1, W(x_2, y)))}{W_y(x_2, y) W_y(x_1, W(x_2, y)) W_y(x_0, W(x_1, W(x_2, y)))}, \\ &= \frac{\alpha_{2,x_0}(x_0, x_1, x_2)}{\beta_2(x_0, x_1, x_2)} + \frac{\beta_{1,x_0}(x_0, x_1, x_2)}{\beta_2(x_0, x_1, x_2)} y. \end{aligned} \tag{66}$$

We want to simplify the expression $J(x_0, x_1, x_2, y)$. First, using Equation (58), we have for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$:

$$J(x_0, x_1, x_2, y) = \frac{\alpha_0'(x_0)}{\beta_0} \frac{1}{W_y(x_2, y)} \frac{1}{W_y(x_1, W(x_2, y))}.$$

From Equation (65), we obtain for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$:

$$J(x_0, x_1, x_2, y) = \frac{\alpha_0'(x_0)}{\beta_0 \kappa_0^2} \left(\frac{\mu_0 \alpha_0(x_2)}{\beta_0} + \nu_0 + \mu_0 y \right)^{1 - \frac{1}{\mu_0}} \left(\frac{\mu_0 \alpha_0(x_1)}{\beta_0} + \nu_0 + \mu_0 W(x_2, y) \right)^{1 - \frac{1}{\mu_0}}.$$

Plugging the expression (64) of $W(x_2, y)$, we deduce that for all $(x_0, x_1, x_2, y) \in [\underline{c}, \overline{c}]^3 \times [0, 1]$, we have:

$$\begin{aligned} J(x_0, x_1, x_2, y) = {} & \frac{\alpha_0'(x_0)}{\beta_0 \kappa_0^2} \left(\frac{\mu_0 \alpha_0(x_2)}{\beta_0} + \nu_0 + \mu_0 y \right)^{1 - \frac{1}{\mu_0}} \\ & \times \left(\frac{\mu_0 \alpha_0(x_1)}{\beta_0} + \nu_0 + \mu_0 \kappa_0 \left(\frac{\mu_0 \alpha_0(x)}{\beta_0} + \nu_0 + \mu_0 y \right)^{\frac{1}{\mu_0}} \right)^{1 - \frac{1}{\mu_0}}. \end{aligned} \tag{67}$$

Equation (66) states that (67) should be an affine function of $y \in [0, 1]$. Since $\mu_0 \neq 0$, it imposes either $\alpha_0'(x) = 0$ for all $x \in [\underline{c}, \overline{c}]$, or $\frac{1}{\mu_0} = 0$, or $\mu_0 = 1$. We can rule out the two first cases (the first one corresponds to $W_x(x, y) = 0$ for all (x, y), which is not possible and the second one corresponds to $W(x, y) = \kappa_0$ for all (x, y) that we can also discard). We deduce from (64) the following expression for the aggregator W for all $(x, y) \in [\underline{c}, \overline{c}] \times [0, 1]$:

$$W(x, y) = \widetilde{\nu}_0 + \widetilde{\alpha}_0(x) + \kappa_0 y,$$

where $\widetilde{\nu}_0 \in \mathbb{R}$ and $\widetilde{\alpha}_0 : [\underline{c}, \overline{c}] \to [0, 1]$, which would imply that $W_{yy} = 0$ which we had ruled out.

A.3 Sufficient conditions

We now show that if we have a KP recursive utility function that admits the aggregator

$$W(x,y) = \alpha(x) + \beta(x)y$$

or the aggregator

$$W(x,y) = \frac{1 - e^{-k(1-\beta)u(x)}(1-\widetilde{k}y)^{\beta}}{\widetilde{k}}$$

then it fulfills ordinal dominance. By definition the KP recursive utility function fulfills

$$U(c,m) = W(c, E[U([m])])$$

If $W(x,y) = \alpha(x) + \beta(x)y$ the agent is an expected utility maximizer (see Corollary 3 in Kreps and Porteus, 1978) and preferences fulfill ordinal dominance.

It remains to consider the case where:

$$W(x,y) = \frac{1 - e^{-k(1-\beta)u(x)}(1-\widetilde{k}y)^{\beta}}{\widetilde{k}}$$

First, we show (straightforward) that preferences that are generated over deterministic consumption path are given by:

$$U(c_1, ...) = \frac{1 - \exp(-k\frac{1}{(1-\beta)}\sum_{i=1}^{\infty}\beta^{i-1}u(c_i))}{\widetilde{k}}$$

Then the proof can be worked out by recurrence.

B Proof of proposition 2

Consider

$$W(x,y) = \alpha(x) + \beta(x)y \tag{68}$$

where $\alpha(\underline{c}) = 0$ and $\alpha(\overline{c}) + \beta(\overline{c}) = 1$.

Note that for this to define a recursive utility function it must be the case that $\beta(c) < 1$ for all $c > \underline{c}$. Otherwise if $\beta(c) = 1$ we would have

$$U(c_\infty) = \alpha(c_\infty) + \beta(c_\infty)U(c_\infty)$$

which would imply $\alpha(c_\infty) = 0$, and $U(c, \underline{c}_\infty) = 0$ which by monotonicity would give $c = \underline{c}$.

Now, one possibility is to have $\beta = cst$, which leads to the aggregator

$$W(x,y) = \alpha(x) + \beta y$$

which is comparable in terms of risk aversion with preferences associated to the aggregator

$$W(x,y)=\frac{1-e^{-k\alpha(x)}(1-\widetilde{k}y)^{\beta}}{\widetilde{k}}$$

Assume that β is not constant. We are going to show that there is no different aggregators that fulfill the ordinal dominance and gives the same ordinal preferences.

We first state the following simple result:

Lemma 4 *Assume that $W_1(x,y)$ and $W_2(x,y)$ are two (well normalized) aggregator representing the same preferences over deterministic consumption paths. Then there exist an increasing function ψ such that $\psi(0)=0,\psi(1)=1$ and*

$$\psi W_1(x,y)=W_2(x,\psi(y))$$

Proof. Consider U_1 and U_2 the utility functions. We have $U_2=\psi(U_1)$ for some increasing function ψ such that $\psi(0)=0,\psi(1)=1$.

We have:

$$\begin{aligned} U_2(c_0,c_1,...) &= W_2(c_0,U_2(c_1,...)) \\ &= W_2(c_0,\psi(U_1((c_1,...)) \end{aligned}$$

But we also have

$$U_2(c_0,c_1,...)=\psi(U_1(c_0,c_1,...))=\psi(W_1(c_0,U_1(c_1,...)))$$

so for all consumption vector:

$$W_2(c_0,\psi(U_1((c_1,...))=\psi(W_1(c_0,U_1(c_1,...)))$$

Posing $y=U_1(c_1,...)$ which varies in $[0,1]$ we get

$$\psi W_1(c_0,y)=W_2(c_0,\psi(y))$$

■

QED.

We can now prove that if $\beta(x)$ is not constant, preferences admitting the aggregator $W(x,y)=\alpha(x)+\beta(x)y$ cannot provide the same ranking over D_0than others preferences fulfilling ordinal dominance. We need therefore to investigate the cases where we can have ϕ such

$$\phi\left(\alpha(x)+\beta(x)y\right)=a(x)+b(x)\phi(y)$$

with $\phi(0)=,\phi(1)=1,\quad \alpha(\underline{c})=a(\underline{c})=0$ and $\alpha(\overline{c})+\beta(\overline{c})=a(\overline{c})+b(\overline{c})=1.$

Derivating with respect to y we get

$$\beta(x)\phi'\left(\alpha(x)+\beta(x)y\right)=b(x)\phi'(y)$$

and taking the log derivative we obtain

$$\beta(x)\frac{\phi''}{\phi'}\left(\alpha(x)+\beta(x)y\right)=\frac{\phi''}{\phi'}(y)$$

Either we have $\frac{\phi''}{\phi'}=0$ and preferences are identical.

Or take $g(x)=x\frac{\phi''}{\phi'}$ and assume that this is $\neq 0$. We have

$$\left(\frac{\beta(x)y}{\alpha(x)+\beta(x)y}\right)g\left(\alpha(x)+\beta(x)y\right)=g(y)$$

With β non constant this implies that $\alpha(x)=0$ (and $xg(x)=cste$) which is in contradiction with $\alpha(\overline{c})+\beta(\overline{c})=1$.

C Proof of Proposition 5

We first provide a technical result:

Lemma 5 *Consider a decreasing function and positive function h and an increasing and positive function b and random variable $\widetilde{x}$. Then*

$$E[h(\widetilde{x})e^{-k\widetilde{x}}]E[b(\widetilde{x})e^{-k\widetilde{x}}]-E[b(\widetilde{x})h(\widetilde{x})e^{-k\widetilde{x}}]E[e^{-k\widetilde{x}}]\geq 0$$

Moreover if $h(x)e^{-kx}$ is decreasing:

$$E[b(\widetilde{x})h(\widetilde{x})e^{-k\widetilde{x}}]E[h^2(\widetilde{x})e^{-2k\widetilde{x}}]-E[b(\widetilde{x})h^2(\widetilde{x})e^{-2k\widetilde{x}}]E[h(\widetilde{x})e^{-k\widetilde{x}}]\geq 0$$

Proof. Denote by $f(x)$ the density function of x. Define $g(x)=e^{-kx}f(x)$ and denote

$$\Delta(a)=\int_{-\infty}^{a}h(x)g(x)dx\int_{-\infty}^{a}b(x)g(x)dx-\int_{-\infty}^{a}b(x)h(x)g(x)dx\int_{-\infty}^{a}g(x)dx$$

We want to show that $\Delta(+\infty)\geq 0$.

We have

$$\Delta'(a)=g(a)\int_{-\infty}^{a}g(x)(h(x)-h(a))(b(a)-b(x))dx\geq 0$$

As $\Delta(-\infty)=0$ we obtain $\Delta(+\infty)\geq 0$, which proves the first point.

Now for the second point we need to show that

$$E[b(\widetilde{x})h(\widetilde{x})e^{-k\widetilde{x}}]E[h^2(\widetilde{x})e^{-2k\widetilde{x}}]-E[b(\widetilde{x})h^2(\widetilde{x})e^{-2k\widetilde{x}}]E[h(\widetilde{x})e^{-k\widetilde{x}}]\geq 0$$

We now define $\gamma(x) = e^{-kx} f(x)h(x)$

$$\Omega(a) = \int_{-\infty}^{a} h(x)e^{-kx}\gamma(x)dx \int_{-\infty}^{a} b(x)\gamma(x)dx - \int_{-\infty}^{a} b(x)e^{-kx}h(x)\gamma(x)dx \int_{-\infty}^{a} \gamma(x)dx$$

We have

$$\begin{aligned}
\Omega'(a) &= \gamma(a)\int_{-\infty}^{a} (b(x)h(a)e^{-ka} + b(a)h(x)e^{-kx} - b(a)h(a)e^{-ka} - b(x)e^{-kx}h(x))\gamma(x)dx \\
&= \gamma(a)\int_{-\infty}^{a} (h(a)e^{-ka} - h(x)e^{-kx})(b(x) - b(a))\gamma(x)dx
\end{aligned}$$

and thus $\Omega'(a) \geq 0$. Since $\Omega(-\infty) = 0$ we obtain $\Omega(+\infty) \geq 0$, which ends proving the lemma.

■

Now to prove Proposition 5 we start from:

$$\frac{1}{R_t} = \frac{1}{\beta u'(c_t)} \frac{E[u'(c_{t+1})\exp(-kU_{t+1})]}{E[\exp(-kU_{t+1})]}$$

we compute

$$\frac{\partial R_t}{\partial k} = \frac{E[\frac{\partial(kU_{t+1})}{\partial k} u'(c_{t+1})\exp(-kU_{t+1})]E[\exp(-kU_{t+1})] - E[u'(c_{t+1})\exp(-kU_{t+1})]E[\frac{\partial(kU)}{\partial k}\exp(-kU_{t+1})]}{E[\exp(-kU_{t+1})]^2}$$

However,

$$\frac{\partial\left(kU_{t+1}\right)}{\partial k} = (1-\beta)u(c_{t+1}) + \beta\frac{E_t[U_{t+2}e^{-kU_{t+2}}]}{E_t[e^{-kU_{t+2}}]}$$

Thus, according to the assumptions of Proposition 5, $\frac{\partial(kU)}{\partial k}$ and U_{t+1} are comonotonic, while U_{t+1} and $u'(c_{t+1})$ are anticomonotonic (since u' is decreasing). We thus have $\frac{\partial(kU_{t+1})}{\partial k} = b(U_{t+1})$ for some increasing function b and $u'(c_{t+1}) = h(U_{t+1})$ for some decreasing function h. A straightforward application of Lemma 5 implies then that $\frac{\partial R_t}{\partial k} < 0$.

With respect to the market price of risk, we have:

$$m_{t,t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}\frac{\exp(-kU_{t+1})}{E[\exp(-kU_{t+1})]}$$

and

$$\left(\frac{\sigma(m_{t,t+1})}{E[m_{t,t+1}]}\right)^2 = \frac{E\left[(u'(c_{t+1}))^2\exp(-2kU_{t+1})\right]}{E\left[(u'(c_{t+1}))\exp(-kU_{t+1})\right]^2} - 1$$

so that

$$\begin{aligned}
\left(\frac{\sigma(m_{t,t+1})}{E[m_{t,t+1}]}\right)\frac{\partial}{\partial k}\left(\frac{\sigma(m_{t,t+1})}{E[m_{t,t+1}]}\right) &= \frac{E\left[(u'(c_{t+1}))\frac{\partial(kU_{t+1})}{\partial k}\exp(-kU_{t+1})\right]E\left[(u'(c_{t+1}))^2\exp(-2kU_{t+1})\right]}{E\left[(u'(c_{t+1}))\exp(-kU_{t+1})\right]^3} \\
&\quad - \frac{E\left[\frac{\partial(kU_{t+1})}{\partial k}(u'(c_{t+1}))^2\exp(-2kU_{t+1})\right]E\left[(u'(c_{t+1}))\exp(-kU_{t+1})\right]}{E\left[(u'(c_{t+1}))\exp(-kU_{t+1})\right]^3}
\end{aligned}$$

Again, since $\frac{\partial(kU_{t+1})}{\partial k} = b(U_{t+1})$ for some increasing function b and $u'(c_{t+1}) = h(U_{t+1})$ for some decreasing function h, a direct application of Lemma 5 implies that

$$\frac{\partial}{\partial k}\left(\frac{\sigma(m_{t,t+1})}{E[m_{t,t+1}]}\right) > 0$$

D Approximate solution to the risk sharing problem

Following Samuelson (1970), we solve the model thanks to an approximation in σ. For any variable x_t, we denote its second order approximation in σ as follows:

$$x_t = x_0 + x_{1,t}\sigma + \frac{1}{2}x_{2,t}\sigma^2 + O(\sigma^3),$$

where x_0 is the constant value of the variable x_t in a certain world.

The Euler equation (13) becomes:

$$\beta E_t \left[\left(d_0 + \sigma\varepsilon^d_{t+1} + \frac{1}{2}\sigma^2 \left(\varepsilon^d_{t+1}\right)^2 \right) \frac{u'(c^i_0 + c^i_{1,t+1}\sigma + \frac{1}{2}c^i_{2,t+1}\sigma^2)}{u'(c^i_0 + c^i_{1,t}\sigma + \frac{1}{2}c^i_{2,t}\sigma^2)} e^{-k_i\left(V^i_{1,t+1}\sigma + \frac{1}{2}V^i_{2,t+1}\sigma^2\right)} \right] \tag{69}$$
$$= \left(p_0 + p_{1,t}\sigma + \frac{1}{2}p_{2,t}\sigma^2 \right) E_t \left[e^{-k_i\left(V^i_{1,t+1}\sigma + \frac{1}{2}V^i_{2,t+1}\sigma^2\right)} \right].$$

We deduce:

$$p_0 = \beta d_0, \tag{70}$$

and at the first order in σ, Equation (69) becomes:

$$\beta E_t \left[\left(d_0 + \sigma\varepsilon^d_{t+1}\right) \left(1 + \left(c^i_{1,t+1} - c^i_{1,t}\right)\sigma \frac{u''(c^i_0)}{u'(c^i_0)} \right) \left(1 - k_i V^i_{1,t+1}\sigma\right) \right] = \left(p_0 + p_{1,t}\sigma\right) E_t \left[1 - k_i V^i_{1,t+1}\sigma\right] + O(\sigma^2),$$

which simplifies after some manipulations into:

$$\beta E_t \left[d_0 + \sigma\varepsilon^d_{t+1} + d_0\left(c^i_{1,t+1} - c^i_{1,t}\right)\sigma \frac{u''(c^i_0)}{u'(c^i_0)} - d_0 k_i V^i_{1,t+1}\sigma \right] = p_0 + p_{1,t}\sigma - E_t\left[p_0 k_i V^i_{1,t+1}\sigma\right] + O(\sigma^2).$$

We deduce:

$$p_{1,t} = \beta d_0 E_t \left[\left(c^i_{1,t+1} - c^i_{1,t}\right) \frac{u''(c^i_0)}{u'(c^i_0)} \right] + O(\sigma). \tag{71}$$

By the same token, we have:

$$q_0 = \beta, \tag{72}$$

$$q_{1,t} = \beta E_t \left[\left(c^i_{1,t+1} - c^i_{1,t}\right) \frac{u''(c^i_0)}{u'(c^i_0)} \right] + O(\sigma), \tag{73}$$

We deduce:

$$q_{1,t} = \frac{p_{1,t}}{d_0}, \tag{74}$$

$$E\left[c^i_{1,t+1}\right] = c^i_{1,t} + \frac{p_{1,t}}{\beta d_0}\frac{u'(c^i_0)}{u''(c^i_0)}. \tag{75}$$

We obtain by difference of Euler equations (13) and (14):

$$\beta E_t \left[\left(\frac{d_{t+1}}{d_0} - 1 \right) \frac{u'(c^i_{t+1})}{u'(c^i_t)} e^{-k_1 V^i(a^i_t, b^i_t)} \right] = \left(\frac{p_t}{d_0} - q_t \right) E_t \left[e^{-k_i V^i(a^i_t, b^i_t)} \right] \tag{76}$$

Since $\frac{d_{t+1}}{d_0}-1$ and $\frac{p_t}{d_0}-q_t$ are $O(\sigma)$, we only need to develop at the first order in σ other terms to obtain a second order approximation of (76):

$$\beta E_t\left[\frac{\sigma\varepsilon^d_{t+1}+\frac{1}{2}\sigma^2\left(\varepsilon^d_{t+1}\right)^2}{d_0}\left(1+\left(c^i_{1,t+1}-c^i_{1,t}\right)\sigma\frac{u''(c^i_0)}{u'(c^i_0)}\right)\left(1-k_iV^i_{1,t+1}\sigma\right)\right]=\left(\frac{p_{2,t}}{d_0}-q_{2,t}\right)\sigma^2+O(\sigma^3)$$

$$\beta E_t\left[\frac{1}{2}\frac{\left(\varepsilon^d_{t+1}\right)^2}{d_0}+\frac{\varepsilon^d_{t+1}}{d_0}\left(\left(c^i_{1,t+1}-c^i_{1,t}\right)\frac{u''(c^i_0)}{u'(c^i_0)}-k_iV^i_{1,t+1}\right)\right]=\frac{p_{2,t}}{d_0}-q_{2,t}$$

By sum and difference between both agents $i=A,B$:

$$\beta E_t\left[\frac{\varepsilon^d_{t+1}}{d_0}\left(c^A_{1,t+1}\frac{u''(c^A_0)}{u'(c^A_0)}-c^B_{1,t+1}\frac{u''(c^B_0)}{u'(c^B_0)}-\left(k_AV^A_{1,t+1}-k_BV^B_{1,t+1}\right)\right)\right]=0 \tag{77}$$

$$\frac{\beta}{2}E_t\left[\frac{\left(\varepsilon^d_{t+1}\right)^2}{d_0}+\frac{\varepsilon^d_{t+1}}{d_0}\left(c^A_{1,t+1}\frac{u''(c^A_0)}{u'(c^A_0)}+c^B_{1,t+1}\frac{u''(c^B_0)}{u'(c^B_0)}-\left(k_AV^A_{1,t+1}+k_BV^B_{1,t+1}\right)\right)\right]=\frac{p_{2,t}}{d_0}-q_{2,t} \tag{78}$$

Computation of the asset holding a^A_0. We simplify Equation (77) in order to obtain an expression of the risky asset holding a^A_0.

We start with the value function expression $V^i_t=(1-\beta)u(c^i_t)-\frac{\beta}{k_i}\log E\left[e^{-k_iV^i_{t+1}}\right]$ (See (11)) and we have at the first order in σ:

$$V^i_0+V^i_{1,t}\sigma=(1-\beta)u(c^i_0)+(1-\beta)c^i_{1,t}u'(c^i_0)\sigma+\beta V^i_0-\frac{\beta}{k_i}\log E\left[1-k_iV^i_{1,t+1}\sigma\right]+O(\sigma^2)$$

$$V^i_0=u(c^i_0)$$

$$\begin{aligned}V^i_{1,t}&=(1-\beta)c^i_{1,t}u'(c^i_0)+\beta E_t\left[V^i_{1,t+1}\right]\\&=(1-\beta)u'(c^i_0)\sum_{k=0}^{\infty}\beta^kE_t\left[c^i_{1,t+k}\right]\end{aligned} \tag{79}$$

We now develop at the first order the budget constraint (12):

$$c^i_{1,t}=\varepsilon^y_t+\varepsilon^d_ta^i_0+d_0\left(a^i_{1,t-1}-\beta a^i_{1,t}\right)+\left(b^i_{1,t-1}-\beta b^i_{1,t}\right)-p_{1,t}a^i_0-q_{1,t}b^i_0$$

$$\sum_{k=0}^{\infty}\beta^kc^i_{1,t+k}=\sum_{k=0}^{\infty}\beta^k\left(\varepsilon^y_{t+k}+\varepsilon^d_{t+k}a^i_0\right)+d_0a^i_{1,t-1}+b_{1,t-1}-\Big(\underbrace{d_0a^i_0+b^i_0}_{=0}\Big)\sum_{k=0}^{\infty}q_{1,t+k}$$

$$\sum_{k=0}^{\infty}\beta^kE_t\left[c^i_{1,t+k}\right]=\left(\varepsilon^y_t+\varepsilon^d_ta^i_0\right)+d_0a^i_{1,t-1}+b^i_{1,t-1} \tag{80}$$

Indeed, the total wealth $d_0a^i_0+b^i_0$ of an agent in an economy without risk is null.

But from (75), we also have $E_t\left[c^i_{1,t+1}\right]=c^i_{1,t}+\frac{q_{1,t}}{\beta}\frac{u'(c^i_0)}{u''(c^i_0)}$, such that $E_t\left[c^i_{1,t+k}\right]=c^i_{1,t}+$

$\frac{1}{\beta}\frac{u'(c_0^i)}{u''(c_0^i)}\sum_{j=0}^{k-1} E_t\left[q_{1,t+j}\right]$. Equation (80) simplifies into:

$$\begin{aligned} \frac{c_{1,t}^i}{1-\beta} + \frac{u'(c_0^i)}{u''(c_0^i)}\sum_{k=1}^{\infty}\beta^{k-1}\sum_{j=0}^{k-1}E_t\left[q_{1,t+j}\right] &= \varepsilon_t^y + \varepsilon_t^d a_0^i + d_0 a_{1,t-1}^i + b_{1,t-1}^i \\ \frac{c_{1,t}^i}{1-\beta} + \frac{u'(c_0^i)}{u''(c_0^i)}\sum_{k=0}^{\infty}\frac{\beta^k}{1-\beta}E_t\left[q_{1,t+k}\right] &= \varepsilon_t^y + \varepsilon_t^d a_0^i + d_0 a_{1,t-1}^i + b_{1,t-1}^i \end{aligned} \tag{81}$$

Therefore, remarking that $c_0^A = c_0^B = c_0$, we can simplify the term in $c_{1,t+1}^i$ of (77) using (81) for $i = A, B$:

$$c_{1,t+1}^A\frac{u''(c_0^A)}{u'(c_0^A)} - c_{1,t+1}^B\frac{u''(c_0^B)}{u'(c_0^B)} = (1-\beta)\frac{u''(c_0)}{u'(c_0)}\left(\left(\left(\varepsilon_{t+1}^y + \varepsilon_{t+1}^d a_0^A\right) + d_0 a_{1,t}^A + b_{1,t}^A\right) - \left(\left(\varepsilon_{t+1}^y + \varepsilon_{t+1}^d a_0^B\right) + d_0 a_{1,t}^B + b_{1,t}^B\right)\right) \tag{82}$$

Analogously using (81), we can simplify the expression (79) of $V_{1,t}^i$ as follows:

$$\begin{aligned} V_{1,t}^i &= (1-\beta)u'(c_0)\sum_{k=0}^{\infty}\beta^k E_t\left[c_{1,t+k}^i\right] \\ &= (1-\beta)u'(c_0)\left(\frac{c_{1,t}^i}{1-\beta} + \frac{u'(c_0)}{u''(c_0)}\sum_{k=0}^{\infty}\frac{\beta^k}{1-\beta}E_t\left[q_{1,t+k}\right]\right) \\ &= (1-\beta)u'(c_0)\left(\varepsilon_t^y + \varepsilon_t^d a_0^i + d_0 a_{1,t-1}^i + b_{1,t-1}^i\right) \end{aligned} \tag{83}$$

We can simplify the term in $V_{1,t+1}^i$ of (77) using (83) for $i = A, B$:

$$k_A V_{1,t+1}^A - k_B V_{1,t+1}^B = (1-\beta)u'(c_0)\left(k_A\left(\varepsilon_{t+1}^y + \varepsilon_{t+1}^d a_0^A + d_0 a_{1,t}^A + b_{1,t}^A\right) - k_B\left(\varepsilon_{t+1}^y + \varepsilon_{t+1}^d a_0^B + d_0 a_{1,t}^B + b_{1,t}^B\right)\right) \tag{84}$$

Substituting expressions (82) and (84) into Equation (77), we deduce remarking that $E_t\left[\varepsilon_{t+1}^d\left(d_0 a_{1,t}^i + b_{1,t}^i\right)\right] = \left(d_0 a_{1,t}^i + b_{1,t}^i\right)E_t[\varepsilon_{t+1}^d] = 0$ and that $a_0^B = -a_0^A$:

$$\begin{aligned} 0 &= E_t\left[\varepsilon_{t+1}^d\left((1-\beta)\frac{u''(c_0)}{u'(c_0)}\varepsilon_{t+1}^d\left(a_0^A - a_0^B\right) - (1-\beta)u'(c_0)\left(k_A\left(\varepsilon_{t+1}^y + \varepsilon_{t+1}^d a_{ss}^A\right) - k_B\left(\varepsilon_{t+1}^y + \varepsilon_{t+1}^d a_{ss}^B\right)\right)\right)\right] \\ 0 &= 2a_0^A\frac{u''(c_0)}{u'(c_0)}E_t\left[\varepsilon_{t+1}^d\varepsilon_{t+1}^d\right] - u'(c_0)E_t\left[\varepsilon_{t+1}^d\left(\left(k_A - k_B\right)\varepsilon_{t+1}^y + a_{ss}^A\left(k_A + k_B\right)\varepsilon_{t+1}^d\right)\right] \end{aligned}$$

We finally deduce the expression of the asset holding of agent A:

$$a_0^A = -\frac{\frac{k_A - k_B}{2}}{\frac{k_A + k_B}{2} - \frac{u''(c_0)}{(u'(c_0))^2}}\frac{E_t\left[\varepsilon_{t+1}^d\varepsilon_{t+1}^y\right]}{E_t\left[\left(\varepsilon_{t+1}^d\right)^2\right]}. \tag{85}$$

If both shocks commove positively, the more risk averse agent holds less risky asset than the more risk averse one. This is in line with the fact that the asset in that case is a bad hedge against the income risk.

Excess return. We now express the excess return in Equation (78):

$$\frac{p_{2,t}}{d_{ss}} - q_{2,t} = \frac{\beta}{2} E_t \left[\frac{\left(\varepsilon^d_{t+1}\right)^2}{d_0} + \frac{\varepsilon^d_{t+1}}{d_0} \left(\frac{u''(c_0)}{u'(c_0)} \left(c^A_{1,t+1} + c^B_{1,t+1}\right) - \left(k_A V^A_{1,t+1} + k_B V^B_{1,t+1}\right) \right) \right]$$

We need to express the terms in $c^i_{1,t+1}$ and $V^i_{1,t+1}$ in (78).

We start with terms in $c^i_{1,t+1}$. Summing Equation (81) for $i = A, B$, we deduce:

$$\frac{1}{2} \frac{c^A_{1,t} + c^B_{1,t}}{1-\beta} = \varepsilon^y_t - \frac{u'(c_0)}{u''(c_0)} \sum_{k=0}^{\infty} \frac{\beta^k}{1-\beta} E_t \left[q_{1,t+k} \right] \tag{86}$$

and:

$$\begin{aligned} \frac{1}{2} \frac{E_t \left[c^A_{1,t+1} + c^B_{1,t+1}\right] - \left(c^A_{1,t} + c^B_{1,t}\right)}{1-\beta} &= -\varepsilon^y_t + \frac{u'(c_0)}{u''(c_0)} \left(\sum_{k=0}^{\infty} \frac{\beta^k}{1-\beta} E_t \left[q_{1,t+k}\right] - \sum_{k=0}^{\infty} \frac{\beta^k}{1-\beta} E_t \left[q_{1,t+1+k}\right] \right) \\ &= -\varepsilon^y_t + \frac{u'(c_0)}{u''(c_0)} \left(\frac{q_{1,t}}{1-\beta} + \sum_{k=1}^{\infty} \frac{\beta^k}{1-\beta} E_t \left[q_{1,t+k}\right] - \sum_{k=1}^{\infty} \frac{\beta^{k-1}}{1-\beta} E_t \left[q_{1,t+k}\right] \right) \end{aligned}$$

Using Equation (75), we can simplify the previous RHS and obtain (after multiplication by β):

$$\begin{aligned} \frac{q_{1,t}}{(1-\beta)} \frac{u'(c^i_0)}{u''(c^i_0)} &= -\beta\varepsilon^y_t + \frac{u'(c_0)}{u''(c_0)} \left(\frac{\beta q_{1,t}}{1-\beta} - \sum_{k=1}^{\infty} \beta^k E_t \left[q_{1,t+k}\right] \right) \\ &= -\beta\varepsilon^y_t + \frac{u'(c_0)}{u''(c_0)} \left(\frac{q_{1,t}}{1-\beta} - \sum_{k=0}^{\infty} \beta^k E_t \left[q_{1,t+k}\right] \right) \end{aligned}$$

We deduce that Equation (86) becomes:

$$\begin{aligned} \frac{1}{2} \frac{c^A_{1,t} + c^B_{1,t}}{1-\beta} &= \varepsilon^y_t - \frac{u'(c_0)}{u''(c_0)} \sum_{k=0}^{\infty} \frac{\beta^k}{1-\beta} E_t \left[q_{1,t+k}\right] \\ &= \frac{1}{1-\beta} \varepsilon^y_t \end{aligned} \tag{87}$$

We now express the terms in $V^i_{1,t+1}$ in (78). Summing (83) for $i = A, B$, we obtain:

$$k_A V^A_{1,t+1} + k_B V^B_{1,t+1} = (1-\beta) u'(c_0) \left((k_A + k_B) \varepsilon^y_t + (k_A - k_B) \varepsilon^d_t a^A_0 + (k_A - k_B) \left(d_0 a^A_{1,t} + b^A_{1,t} \right) \right) \tag{88}$$

Plugging (87) and (88) into (78), we obtain:

$$\begin{aligned} \frac{p_{2,t}}{d_0} - q_{2,t} - \frac{\beta}{2} E_t \left[\frac{\left(\varepsilon^d_{t+1}\right)^2}{d_0} \right] &= \beta \frac{u''(c_0)}{u'(c_0)} E_t \left[\frac{\varepsilon^d_{t+1} \varepsilon^y_{t+1}}{d_0} \right] \\ &\quad - \frac{\beta}{2} (1-\beta) u'(c_0) E_t \left[\frac{\varepsilon^d_{t+1} \varepsilon^y_{t+1}}{d_0} \left((k_A + k_B) - \frac{\frac{(k_A - k_B)^2}{2}}{\frac{k_A + k_B}{2} - \frac{u''(c_0)}{(u'(c_0))^2}} \right) \right] \end{aligned}$$

We finally obtain:

$$-\left(\frac{p_{2,t}}{d_0} - q_{2,t} - \frac{\beta}{2} E_t \left[\frac{\left(\varepsilon^d_{t+1}\right)^2}{d_0} \right] \right) = \beta E_t \left[\frac{\varepsilon^d_{t+1} \varepsilon^y_{t+1}}{d_0} \right] \left(-\frac{u''(c_0)}{u'(c_0)} + \frac{(1-\beta) u'(c_0)}{2} \left((k_A + k_B) - \frac{\frac{(k_A - k_B)^2}{2}}{\frac{k_A + k_B}{2} - \frac{u''(c_0)}{(u'(c_0))^2}} \right) \right).$$

This provide the expression of the excess return.