WHEN IS AMBIGUITY-ATTITUDE CONSTANT?*

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Abstract

This paper studies how updating affects ambiguity-attitude. In particular we focus on the generalized Bayesian update of the Jaffray-Phillipe sub-class of Choquet Expected Utility preferences. We find conditions for ambiguity-attitude to be the same before and after updating. A necessary and sufficient condition for ambiguity-attitude to be unchanged when updated on an arbitrary event is for the capacity to be neo-additive. We find a condition for updating on a given partition to preserve ambiguity-attitude. We relate this to necessary and sufficient conditions for dynamic consistency. Finally we study whether ambiguity increases or decreases after updating.

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1 INTRODUCTION

This paper studies how to update ambiguous beliefs as new information arrives. There have been a number of previous papers on updating and ambiguity, for instance, Eichberger and Kelsey (1996), Epstein and Schneider (2003), or Sarin and Wakker (1998). Much of this literature has explicitly or implicitly assumed ambiguity-aversion. Less attention has been paid to the problem of updating preferences which are not necessarily ambiguity-averse. However there is substantial experimental evidence that individuals are not uniformly ambiguity-averse but also at times display ambiguity-seeking, (Abdellaoui, Vossmann, and Weber (2005), Kilka and Weber (2001) and Wu and Gonzalez (1999)).

We shall use the Choquet expected utility (henceforth CEU) model of ambiguity, see Schmeidler (1989). CEU represents the decision-maker's beliefs by a capacity or non-additive belief. Preferences are represented by the expected value of a utility function with respect to this capacity. The expectation is expressed as a Choquet integral, Choquet (1953-4). When new information is received we shall assume the decision-maker updates his/her capacity but does not change the utility function or the form of the CEU functional. We think this is reasonable since in our opinion the capacity is the only part of the CEU functional which reflects the decision-maker's subjective perception of the environment. The other aspects of the representation are personal characteristics of the decision-maker.

At present there is no general agreement how to update a capacity. It is desirable that the procedure should coincide with Bayesian updating when the capacity is additive, i.e. there is no ambiguity. However a number of ways of updating capacities have been proposed which have this property, (see section 2.2). We believe the most promising is the Generalized Bayesian Updating rule (henceforth GBU), Eichberger, Grant, and Kelsey (2007) and Horie (2007). (See Section 3.1 for a definition of the GBU update.)

Chateauneuf, Eichberger, and Grant (2007) axiomatized CEU preferences where beliefs are represented by a class of capacities known as neo-additive capacities. In this case preferences can be represented as maximizing a weighted average of the best pay-off, the worst pay-off and an average pay-off. That is, the Choquet integral with respect to the neo-additive capacity ν of the state-utility vector $u \circ a$ associated with the act a (a mapping from states to outcomes) may be expressed as:

$$\int u(a(s))\,d\nu(s) = \delta\alpha \min_{s\in S} u(a(s)) + \delta(1-\alpha)\max_{s\in S} u(a(s)) + (1-\delta)\cdot \mathbf{E}_\pi u(a(s)), \quad (1)$$

where π is a probability measure on S and $\mathbf{E}_\pi$ denotes conventional expectation with respect to π.

In Eichberger, Grant, and Kelsey (2010), we showed the GBU update of a neo-additive capacity is also neo-additive and has the same ambiguity-attitude, i.e. the same value of α. We find this a particularly appealing and intuitive property. Firstly it implies that the class of neo-additive preferences is closed under GBU updating. Secondly, notice that if ν is a neo-additive capacity given by $\nu(E) \equiv (1-\delta)\pi(E) + \delta(1-\alpha)$, it may be expressed as the convex combination $\alpha\mu + (1-\alpha)\bar{\mu}$, where $\mu(E) \equiv (1-\delta)\pi(E)$ is a convex capacity and $\bar{\mu}(E) \equiv 1-\mu(E^c) = (1-\delta)\pi(E)+\delta$ is its dual. Since μ depends on the beliefs represented by π and the ambiguity which the individual perceives as embodied in the parameter δ, we argue it reflects a subjective description of the environment and as such it is reasonable that μ (and hence its conjugate $\bar{\mu}$) should be revised when new information is received. In contrast α represents attitude of the individual towards the ambiguity and as such it is desirable that this be a characteristic of the individual that is invariant to the receipt of new information.

This paper investigates whether there is a larger class of capacities which have this desirable property. It can be argued that the CEU model is too general since it is exponential in the number of states. For the analysis to be tractable, we restrict attention to prior beliefs in the form of Jaffray-Phillipe (henceforth JP)-capacities (Jaffray and Philippe (1997)). These also have the advantage that they allow for a clean separation between an individual's perception of the ambiguity she faces and her attitudes towards it. These are capacities which may be written in the form, $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where μ is a convex capacity, $\bar{\mu}$ is its dual and $0 \leqslant \alpha \leqslant 1$. The Choquet integral with respect to a JP-capacity has the following representation

$$\int u(a(s))\,d\nu(s) = \alpha \min_{\pi\in\mathcal{C}(\mu)} \mathbf{E}_\pi u(a(s)) + (1-\alpha)\max_{\pi\in\mathcal{C}(\mu)} \mathbf{E}_\pi u(a(s)),$$

where $C(\mu)$ is a closed and convex set of probability distributions on S corresponding to the core of the capacity μ, that we shall interpret as the individual's perception of ambiguity, and where α is a measure of her degree of (relative) pessimism towards ambiguity.[1]

First we shall consider the problem of updating on an arbitrary event. In this case we require the update of a JP-capacity to have the same ambiguity-attitude when updated on any non-trivial event. We find that, under some mild assumptions, this property will hold if and only if the original preferences can be represented by a neo-additive capacity.[2] This provides a new characterization of neo-additive capacities.

However it can be argued that it is too strong to require updates on *all* events to preserve ambiguity-attitude. It is sufficient to require that ambiguity-attitude is preserved at the events which an individual actually has to make decisions. To model this we consider the case where there is a given partition of the state space. Ex-post it will be revealed in which element of the partition the true state lies. This is a more restrictive problem. However it is one of economic interest. For instance if an individual faces a fixed decision tree then (s)he will only need to update beliefs on events which may be reached in that tree. It is not necessary to consider updates conditional on other events. In practice experimental tests of updating have this form. (Cohen, Gilboa, Jaffray, and Schmeidler (2000)).

Another example is where an individual may get one of a finite number of signals, which give information about the process determining the state. Updating conditional on a signal, is related to updating on a partition as follows. Let S denote the state space and let C be the set of possible signals. For simplicity we shall assume that both S and C are finite. Beliefs can be represented by a prior capacity over the Cartesian product of the state space and the signal space. The signals generate a partition of the product space consisting of sets of the form $S \times c, c \in C$. Henceforth we adopt the terminology appropriate for updating beliefs on signals. However this is for convenience only and does not restrict the generality of the analysis.

For updating on a partition, we find that a larger class of capacities will have ambiguity-attitude preserved by updating. We show that if the GBU-updates have the same ambiguity-attitude as the original preferences, the prior beliefs must lie in a sub-class of JP capacities we

[1] At present there are some unresolved issues concerning how to separate perceptions of ambiguity from ambiguity-attitude. Ghirardato, Maccheroni, and Marinacci (2004) present an alternative way to differentiate between ambiguity and ambiguity attitude. But Eichberger, Grant, and Kelsey (2008) and Eichberger, Grant, Kelsey, and Koshevoy (2010) show that for a finite state space, Ghirardato, Macheroni and Marinacci's separation implies that the capacity cannot exhibit a constant attitude towards ambiguity. Klibanoff, Mukerji, and Seo (2011) and Wakker (2011) present theories of ambiguity and ambiguity-attitude, which are closer to the interpretation in the present paper.

[2] This is close to being a converse to Proposition 1 of Eichberger, Grant, and Kelsey (2010).

refer to as partitionaly-additive JP-capacities, (henceforth PAJP). These capacities are additive over events, which lie in the algebra generated by the partition but may be non-additive over other events.

To understand these results it is helpful to think of ex-ante ambiguity as being derived from two sources. There is ambiguity about which state will be observed. In addition the information which the signals convey may itself be more or less ambiguous. We find that an increase in either source of ambiguity will increase ex-post ambiguity. Moreover a signal which confirms prior beliefs will tend to reduce ambiguity, while an unexpected signal will increase ambiguity. Hence we believe that for this class of capacities, GBU updating has intuitive properties.

We find necessary and sufficient conditions for JP capacities to be dynamically consistent under GBU updating. We show these are only slightly stronger than the conditions for ambiguity-attitude to be invariant when updating. This is intuitive since changes in ambiguity-attitude are likely to be a major cause of dynamic inconsistency.

Organization of the Paper In the next section we present our framework and definitions. Section 3 studies updating on an arbitrary event. Updating with a given partition of events is studied in Section 4 and Section 5 concludes. The appendix contains proofs of those results not proved in the text.

2 FRAMEWORK AND DEFINITIONS

2.1 Choquet Expected Utility

There is a finite set S, of states of nature. There is a set of consequences X, which is assumed to be a convex subset of $\mathbb{R}^n$. An act is a function $a : S \to X$. Let $A(S)$ denotes the space of all acts. The decision-maker has a preference relation $\succcurlyeq$ defined over $A(S)$. We shall assume that $\succcurlyeq$ satisfies the CEU axioms, (Schmeidler (1989) Sarin and Wakker (1992)). The CEU model of ambiguity represents beliefs as capacities. A capacity assigns non-additive weights to subsets of S. Formally, they are defined as follows.

Definition 2.1 *A* capacity *on S is a function $\nu : \mathcal{P}(S) \to \mathbb{R}$ such that $A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$ and $\nu(\varnothing) = 0,\ \nu(S) = 1$.*[3] *The* dual capacity *$\bar{\mu}$ is defined by $\bar{\mu}(A) = 1 - \mu(A^c)$.*

[3] As usual, $\mathcal{P}(S)$ denotes the set of all subsets of S.

The capacity and its dual are two alternative ways of representing the same information. A special case of a capacity is the Hurwicz capacity, defined below.

Definition 2.2 *The Hurwicz capacity* ϕ *on* S *is defined by* $\phi(A) = 0$, *for all* $A \subsetneqq S, \phi(S) = 1$.

In the CEU model, preferences over $A(S)$ are represented by the Choquet expected value of the utility function u, which is defined below.

Notation 2.1 *Since* S *is finite, one can order the utility from a given act* $a : u\left(a^1\right) > u\left(a^2\right) > \ldots > u\left(a^{r-1}\right) > u\left(a^r\right),$ *where* $u\left(a^1\right), \ldots, u\left(a^r\right)$ *are the possible utility levels yielded by action* a. *Denote by* $A_k(a) = \{s \in S |\ u(a(s)) \geqslant u\left(a^k\right)\}$ *the set of states that yield a utility at least as high as* a^k. *By convention, let* $A_o(a) = \emptyset$.

Definition 2.3 *The* Choquet expected utility *of act* a *with respect to capacity* ν *is:*

$$\int u(a(s))d\nu(s) = \sum_{k=1}^{r} u\left(a^k\right)[\nu(A_k(a)) - \nu(A_{k-1}(a))].$$

Definition 2.4 *A capacity,* μ, *is* convex *if* $\mu(A \cup B) \geqslant \mu(A) + \mu(B) - \mu(A \cap B)$.

Convex capacities can be associated in a natural way with a set of probability distributions called the *core* of the capacity.

Definition 2.5 *Let* μ *be a capacity on* S. *The* core, $\mathcal{C}(\mu)$, *is defined by*

$$\mathcal{C}(\mu) = \{p \in \Delta(S); \forall A \subseteq S, p(A) \geqslant \nu(A)\}.$$

Schmeidler (1989) has shown that for a convex capacity, $\int u(a(s))\, d\mu(s) = \min_{p \in \mathcal{C}(\mu)} \mathbf{E}_p u(a(s))$, where $\mathbf{E}_p$ denotes a conventional expectation with respect to the probability p. A stronger condition than convexity is that the capacity be a belief function, defined below.

Definition 2.6 *A capacity* μ *on* S *is a* belief function *if for all* $A_1, ..., A_m \subseteq S$;

$$\mu\left(\bigcup_{i=1}^{m} A_i\right) \geqslant \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \varnothing}} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} A_i\right)$$

for all $m, 2 \leqslant m \leqslant \infty$.

Convexity is the special case where this property is only required to hold for $m = 2$.

Remark 2.1 *Recall that any set function μ on S has a Möbius inverse $\beta : \mathcal{P}(S) \to \mathbb{R}$ with the property that $\mu(A) = \sum_{B \subseteq A} \beta_B$ and $\sum_{B \subseteq S} \beta_B = 1$. One can show that a capacity is a* belief function *if and only if its Möbius inverse is non-negative i.e. for all $B \subseteq S, \beta_B \geqslant 0$, (Dempster (1967), Shafer (1976)).*

2.1.1 Jaffray-Phillipe Capacities

This section introduces the class of JP-capacities which will prove important in our analysis. Jaffray and Philippe (1997) study capacities which may be written as a convex combination of a convex capacity μ and its dual. As we saw in the introduction above, neo-additive capacities are a special case. We shall restrict attention to JP-capacities since there is a natural way to distinguish between the perception of ambiguity and the attitude towards this ambiguity for such capacities. Note that here we only study deviations from expected utility due to ambiguity. In other words we assume that the decision-makers use expected utility for all decisions with known probabilities. JP-capacities are formally defined as follows.

Definition 2.7 *A capacity ν on S is a* JP-capacity *if there exists a convex capacity μ and $\alpha \in [0,1]$, such that $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$.*

We take the degree of ambiguity associated with the JP-capacity to correspond to standard measures of ambiguity for convex capacities.

Definition 2.8 *Let μ be a convex capacity on S. Define the degree of ambiguity of event A associated with the capacity μ by:*

$$\chi(\mu, A) = \bar{\mu}(A) - \mu(A),$$

and the maximal degree of ambiguity associated with μ by

$$\lambda(\mu) = \max\{\chi(\mu, A) : \varnothing \subsetneqq A \subsetneqq S_{-i}\}.$$

This measure provides an upper bound on the amount of ambiguity which the decision-maker perceives.[4] The degree of ambiguity measures the deviation from (binary) additivity. For a probability it is equal to zero. Convex capacities have degrees of ambiguity between 0 and 1, with higher values corresponding to more ambiguity. For a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, we apply this definition to the convex part μ.

As the following proposition shows, the CEU of a JP-capacity is a convex combination of the minimum and the maximum expected utility over the set of probabilities in the core of μ.

Proposition 2.1 *(Jaffray and Philippe (1997)) The CEU of a utility function u with respect to a* JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ *on S is:*

$$\int u(a(s))\, d\nu(s) = \alpha \min_{\pi \in \mathcal{C}(\mu)} \mathbf{E}_\pi u(a(s)) + (1-\alpha) \max_{\pi \in \mathcal{C}(\mu)} \mathbf{E}_\pi u(a(s)).$$

Thus if beliefs may be represented by JP-capacities, preferences lie in the intersection of the CEU and multiple priors models. Proposition 2.1 suggests an interpretation of the parameter α as a degree of (relative) pessimism, since it gives a weight to the worst expected utility an individual could expect from the act a. If $\alpha = 1$, then we obtain a special case of the MEU model axiomatized by Gilboa and Schmeidler (1989). On the other hand, the weight $(1-\alpha)$ given to the best expected utility which a player can obtain with act a provides a natural measure for his/her optimism. For $\alpha = 0$ we have a pure optimist, while in general for $\alpha \in (0,1)$, the player's preferences have both optimistic and pessimistic features. Ambiguity may be measured by the core of the convex capacity μ. A larger core would correspond to a situation, which is perceived to be more ambiguous. Hence JP capacities allow a distinction between ambiguity and ambiguity-attitude.

The neo-additive capacity defined below is a special case of a JP-capacity, which will prove useful in our analysis.

Definition 2.9 *Let α, δ be real numbers such that $0 < \delta < 1, 0 < \alpha < 1$, define a neo-additive-capacity ν on S by $\nu(A) = \delta(1-\alpha) + (1-\delta)\pi(A)$, for $\emptyset \subsetneqq A \subsetneqq S$, where π is an additive probability distribution on S.*

[4] The definition is based on one in Dow and Werlang (1992).

This capacity can be interpreted as describing a situation where the decision maker's 'beliefs' are represented by the probability distribution π. However (s)he has some doubts about these beliefs. This ambiguity about the true probability distribution is reflected by the parameter δ. The highest possible level of ambiguity corresponds to $\delta = 1$, while $\delta = 0$ corresponds to no ambiguity. The reaction to these doubts is in part pessimistic and in part optimistic. As for JP capacities, ambiguity-attitude may be measured by the parameter α. The Choquet expected utility of an act a with respect to the neo-additive-capacity ν is given by equation (1). The Choquet integral for a neo-additive capacity is a weighted average of the highest payoff, the lowest payoff and an average payoff.

2.2 Updating Rules

In this section we describe some procedures for updating preferences as new information is received. As discussed in the introduction we confine attention to procedures which update the capacity while leaving other features of the CEU functional unchanged. Below we define three of the most common rules for updating non-additive beliefs. All of them coincide with Bayesian updating when beliefs are additive.

Definition 2.10 *Let ν be a capacity on S and let $E \subseteq S$. The* Generalised Bayesian Update (henceforth GBU) *of ν conditional on E is given by:*

$$\nu_E(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} = \frac{\nu(A \cap E)}{\nu(A \cap E) + \bar{\nu}(A^c \cap E)}.$$

The GBU update has been axiomatized in Eichberger, Grant, and Kelsey (2007) and Horie (2007).

Definition 2.11 *Let ν be a capacity on S and let $E \subseteq S$. The* Dempster-Shafer update *(henceforth DS-update) of ν conditional on E is given by:*

$$\nu_E(A) = \frac{\nu(A \cup E^c) - \nu(E^c)}{1 - \nu(E^c)} = \frac{\bar{\nu}(E) - \bar{\nu}(A^c \cap E)}{\bar{\nu}(E)}.$$

The DS-update has been axiomatized in Gilboa and Schmeidler (1993), where it is shown to be equivalent to a maximum likelihood updating procedure. The final updating rule we consider is the Optimistic update defined below. This was also axiomatized by Gilboa and Schmeidler (1993). This rule assumes that the worst possible outcome occurred on the complement of E, hence the term optimistic.

Definition 2.12 *Let ν be a capacity on S and let $E \subseteq S$. The* Optimistic update *of ν conditional on E is given by:*

$$\nu_E(A) = \frac{\nu(A \cap E)}{\nu(E)}.$$

In Eichberger, Grant, and Kelsey (2010) we show that when GBU is applied to updating neo-additive capacities, the ambiguity-attitude parameter α is unchanged. Moreover it is the only one of the three updating rules with this property. In the present paper we investigate whether a similar result applies to the larger class of JP-capacities.

3 UPDATING ON AN EVENT

This section finds conditions for the GBU update of a JP-capacity on an arbitrary event to have the same ambiguity-attitude parameter α, as the original capacity. Under the assumption that μ is a belief function we can show that a necessary and sufficient condition for this property to hold is that the original capacity be neo-additive. This provides a new motivation for neo-additive capacities. Finally we consider the DS and Optimistic updates. We show that for these updates ambiguity-attitude is only constant if the prior capacity is additive.

3.1 Generalized Bayesian Updates

The following result finds a necessary and sufficient condition for the update of a JP-capacity to have the JP form with the same ambiguity-attitude parameter.

Lemma 3.1 *Let μ be a given convex capacity on S. Define $\nu^\alpha = \alpha\mu + (1-\alpha)\bar{\mu}$. Consider a given event E. Then a necessary and sufficient condition for the GBU update of ν^α conditional on E to be a JP capacity with the same α, for all α, $0 \leqslant \alpha \leqslant 1$, i.e. $\nu_E^\alpha = \alpha\mu_E + (1-\alpha)\bar{\mu}_E$,*[5] *is that for all partitions A, B of $E, A \cup B = E, A \cap B = \varnothing$:*

$$\mu(A \cup E^c) - \mu(A) = \mu(B \cup E^c) - \mu(B). \tag{2}$$

[5] To clarify we require this equation to hold for *all* $\alpha, 0 \leqslant \alpha \leqslant 1$ but only for the *given* event E. The capacity μ_E depends on E but is independent of α.

Remark 3.1 *A sufficient condition for equation (2) to be satisfied is for all* $F \subseteq E$, $\mu(F \cup E^c) = \mu(F) + \mu(E^c)$. *This condition is not also necessary. However in practice it may be easier to check than the necessary and sufficient condition.*

If we strengthen our assumptions by requiring μ to be a belief function then we can show that a necessary and sufficient condition for the ambiguity-attitude to be the same before and after GBU updating is that the capacity be neo-additive. We provide the converse to Proposition 1 of Eichberger, Grant, and Kelsey (2010) for the case where μ is a belief function.

Proposition 3.1 *Let* $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ *be a JP-capacity where* μ *is a* belief function *on* S, $0 \leqslant \alpha \leqslant 1$ *and* $|S| \geqslant 4$. *Let* ν_E *denote the GBU update of* ν *conditional on* E. *Then a necessary and sufficient condition for* ν_E *to be a JP-capacity with the same* α *for all* $E \subsetneq S$ *is that* ν *be neo-additive.*

In practice the condition $|S| \geqslant 4$ is not restrictive. If there are three or less states then after updating there at most two states will remain possible. If there are only two states, JP-capacities are over-determined. Thus four states is the minimum needed to have a meaningful updating problem. The following example shows that there exists a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, which is not neo-additive even though μ satisfies equation (2). Since μ is convex but not a belief function this demonstrates that it is not possible to drop that requirement in the above result.

Example 3.1 *Suppose there are* 4 *states. The example is a symmetric capacity on* S, *by which we mean that the capacity of an event only depends on the number of states in the event. We adopt the notation that* $\mu(m)$ *denotes the capacity of an event with* m *states. Choose* $\eta < \frac{1}{4}$ *and* $\epsilon < \frac{1}{4} - \eta$. *Let* $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, *where* μ *is the symmetric capacity given by* $\mu(0) = 0, \mu(1) = \eta, \mu(2) = 2\eta + \epsilon, \mu(3) = 3\eta + 2\epsilon$ *and* $\mu(4) = 1$. *Then as Proposition A.1 shows, the updates of* ν *have the JP-form with the same* α, *however* ν *is not neo-additive. The capacity* μ *is convex but is not a belief function.*

3.2 Dempster-Shafer and Optimistic Updates

In Eichberger, Grant, and Kelsey (2010) we found that the DS-update of a neo-additive capacity always displayed ambiguity-aversion no matter what ambiguity-attitude is displayed in the original beliefs. We found the opposite problem with the Optimistic update of a neo-additive capacity, which always displays ambiguity-seeking. We interpreted this result to be a disadvantage of the DS and optimistic updating rules. However another possible interpretation is that these results highlight a restrictive property of neo-additive capacities.

The following results show that for any JP-capacity, the DS and Optimistic updates do not preserve ambiguity-attitude. This is similar to but less extreme than the earlier result for neo-additive capacities reported in Eichberger, Grant, and Kelsey (2010). This strengthens our conclusion that the DS and Optimistic rules are not suitable for capacities which display both ambiguity-aversion in some choices and ambiguity-seeking in others.

Proposition 3.2 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a JP-capacity where $0 \leqslant \alpha \leqslant 1$. Let $\hat{\nu}_E$ denote the DS update of ν conditional on E. Then a necessary and sufficient condition for $\hat{\nu}_E$ to be a JP-capacity with the same α for all $E \subsetneqq S$ is that ν be additive.*

In Eichberger, Grant, and Kelsey (2010) we found that the DS-update of a neo-additive capacity is always convex and thus has an extreme pessimistic bias. Below we investigate whether this result can be generalized to a larger class of capacities. We find that it is not generally true that the DS-update is always convex.

Assume $A \subsetneqq E \subsetneqq S$. Suppose that event $E \subsetneqq S$, is observed. Let $\hat{\nu}_E^{\alpha}(A)$ denote the DS-update conditional on E. Then $\hat{\nu}_E^{\alpha}(A) = \frac{\alpha\mu(A\cup E^c)+(1-\alpha)(1-\mu((A\cup E^c)^c))-\alpha\mu(E^c)-(1-\alpha)(1-\mu(E))}{1-\alpha\mu(E^c)-(1-\alpha)(1-\mu(E))}$. The denominator is just a constant of normalization. Hence we shall focus on the numerator, which is equal to:

$$\mu(A\cup E^c) - (1-\alpha)\mu(A\cup E^c) + (1-\alpha) - (1-\alpha)\mu((A\cup E^c)^c) - 1 + \alpha + \mu(E) - \alpha\mu(E)$$
$$= \mu(A\cup E^c) + (1-\alpha)[1-\mu(A\cup E^c) - \mu((A\cup E^c)^c)] + \alpha[1-\mu(E)-\mu(E^c)] - 1 + \mu(E).$$

The first term is (up to normalization) a convex capacity. The second term is the degree of ambiguity of $\mu(A\cup E^c)$ and the final term is a constant, which has the effect of ensuring $\hat{\nu}_E^{\alpha}$ satisfies the equality $\hat{\nu}_E^{\alpha}(\varnothing) = 0$. Thus the DS-update may be seen as the sum of a convex capacity and a measure of ambiguity of the prior. Define a set function ϕ by $\phi(A) = 1 - \mu(A\cup E^c) - \mu((A\cup E^c)^c)$, i.e. the degree of ambiguity of $\mu(A\cup E^c)$. One can show that convexity of μ implies concavity of ϕ. Thus it is not possible to tell whether $\hat{\nu}_E^{\alpha}$ is convex or concave without additional information. If α is close to 1, the first term will dominate and hence the DS-update will be convex. If the prior is neo-additive, the degree of ambiguity is constant, which again implies the DS-update is convex.

This might lead one to think the DS-update tends to be convex, however the numerator of the DS-update may also be rearranged as follows:

$$1 - \mu\left(\left(A \cup E^c\right)^c\right) + \alpha\mu\left(A \cup E^c\right) - \alpha + \alpha\mu\left(\left(A \cup E^c\right)^c\right) - \alpha\mu\left(E^c\right) - (1-\alpha)\left(1 - \mu(E)\right)$$
$$= \bar{\mu}\left(A \cup E^c\right) - \alpha\left[1 - \mu\left(A \cup E^c\right) - \mu\left(\left(A \cup E^c\right)^c\right)\right] + \alpha\left[1 - \mu(E) - \mu\left(E^c\right)\right] - 1 + \mu(E).$$

This is the sum of a concave capacity and a degree of ambiguity. Hence if α is sufficiently close to 0, then the DS-update is concave. This suggests that the analysis of Eichberger, Grant, and Kelsey (2010) for neo-additive capacities is a special case, which arises because the degree of ambiguity of μ is constant for a neo-additive capacity. In general there is no presumption that the DS-update will be convex.

Next we prove a similar result for the optimistic updating rule.

Proposition 3.3 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a JP-capacity where $0 \leqslant \alpha \leqslant 1$. Let $\bar{\nu}_E$ denote the Optimistic update of ν conditional on E. Then a necessary and sufficient condition for $\bar{\nu}_E$ to be a JP-capacity with the same α for all $E \subsetneqq S$ is that ν be additive.*

Thus we have partially generalized the analysis of Eichberger, Grant, and Kelsey (2010) to JP capacities. Only the GBU update preserves ambiguity-attitude. However in general it is not possible to argue that the Optimistic update expresses ambiguity-preference. The reasoning is similar to that given for the DS-update in the previous section.

4 LEARNING FROM SIGNALS

In this section we consider the problem of updating beliefs when there is a given partition of the state space. As argued in the introduction, the problem of updating on a signal can be interpreted as a special case of updating on a partition. In this context, we find a necessary and sufficient condition for ambiguity-attitude to be the same before and after updating is that the prior capacity lie in a class of capacities we refer to as PAJP capacities (defined below). This restricts the set of capacities which can be used as a prior. The restriction is close to the conditions found by Eichberger, Grant, and Kelsey (2005), for full dynamic consistency.

Let $E_1, ..., E_K$ be a partition of S. There are two time periods $t = 0$ and $t = 1$. The decision-maker has initial beliefs at time $t = 0$. At time $t = 1$, (s)he observes which element of the partition obtains and updates his/her beliefs. We shall use terminology appropriate to the problem of updating on a signal. Thus we shall refer to $E_1, ..., E_K$ as *signals.* However the analysis is applicable more generally.

4.1 Partitionaly-additive JP-Capacities

Below we define a subclass of JP-capacities, which we call partitionaly-additive JP-capacities (PAJP). We then show that a sufficient condition for a capacity to have the same ambiguity-attitude before and after updating on a partition is that it be a PAJP capacity. Under some assumptions we show that this condition is also necessary.

Definition 4.1 *A capacity ν is a* partitionaly-additive JP-capacity *(PAJP) if it has the form $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where μ is a convex capacity defined by:*

$$\mu(D) = (1-\delta)\sum_{k=1}^{K} q_k \mu_k (D \cap E_k), D \subsetneqq S; \qquad \mu(S) = 1, \tag{3}$$

where $0 < \delta < 1, q$ is a probability distribution over $\{E_1, ..., E_K\}$ the elements of the partition and μ_k is a convex capacity on E_k.

The capacity $\mu(D)$ may be viewed as the fraction $(1-\delta)$ of the expected capacity of D according to the capacities μ_k. These are defined on the elements of the partition for which the signal is measurable. Notice that if $\delta = 0$ then $\mu(D)$ is equal to this expectation and the capacity is additive over the partition. If $\delta \in (0,1)$, then for all non-empty events $D \subsetneqq S$, the conjugate capacity $\bar{\mu}$ is given by, $\bar{\mu}(D) = 1 - (1-\delta)\sum_{k=1}^{K} q_k \mu_k (D^c \cap E_k)$ $= \delta + (1-\delta)\sum_{k=1}^{K} q_k \bar{\mu}_k (D \cup E_k^c) = \delta + (1-\delta)\sum_{k=1}^{K} q_k \bar{\mu}_k ((D \cup E_k^c) \cap E_k)$, since $\bar{\mu}_k(E_k) = 1$. Finally $\bar{\mu}(D) = \delta + (1-\delta)\sum_{k=1}^{K} q_k \bar{\mu}_k (D \cap E_k)$. Thus we obtain for all non-empty events $D \subsetneqq S$,

$$\begin{aligned} \nu(D) &= (1-\alpha)\delta + (1-\delta)\sum_{k=1}^{K} q_k \left[\alpha\mu_k (D \cap E_k) + (1-\alpha)\bar{\mu}_k (D \cap E_k)\right] && (4) \\ &= (1-\delta)\sum_{k=1}^{K} q_k \nu_k (D \cap E_k) + \delta(1-\alpha), && (5) \end{aligned}$$

where $\nu_k = \alpha\mu_k + (1-\alpha)\,\bar{\mu}_k$ is, by construction, a JP-capacity on E_k. Further straight forward calculation yields, for all non-empty events $D \subsetneqq S$,

$$\bar{\nu}(D) = (1-\delta)\sum_{k=1}^{K} q_k \bar{\nu}_k (D \cap E_k) + \delta\alpha. \tag{6}$$

Expression (5) can be viewed as saying that the weight assigned to event D by the capacity ν, is a convex combination of the weight assigned by $K+1$ capacities. The expectation of the K JP-capacities ν_k's defined on the elements of the partition on which the signal is measurable is weighted by $(1-\delta)$. But a fraction δ is reserved for the Hurwicz capacity that puts weight α on the worst outcome and weight $(1-\alpha)$ on the best outcome.[6]

4.2 Updating Partitionaly-additive JP-Capacities

The following result finds the GBU update of a PAJP-capacity ν. In particular we see that for each possible realization of the signal the update is a JP capacity with the same α. Thus the effect of updating is to revise μ to take account of the new information, while leaving the ambiguity-attitude unchanged. The convex capacity μ and hence its conjugate $\bar{\mu}$ depend on beliefs and the ambiguity which the individual perceives. Therefore it reflects a subjective description of the environment and as such we argue it is reasonable that μ (and hence its conjugate $\bar{\mu}$) should be revised when new information is received. In contrast α represents the attitude of the individual towards the ambiguity she perceives and as such we argue that this should be invariant to the receipt of new information.

Proposition 4.1 *The GBU update of the PAJP-capacity, ν conditional on event E_k is given by:* $\hat{\nu}_k(A) = \left(1-\hat{\delta}\right)\nu_k(A \cap E_k) + \hat{\delta}(1-\alpha)$

$$= \alpha\left(\frac{(1-\delta)\,q_k\mu_k(A \cap E_k)}{\delta + (1-\delta)\,q_k}\right) + (1-\alpha)\left(1 - \frac{(1-\delta)\,q_k\mu_k(A^c \cap E_k)}{\delta + (1-\delta)\,q_k}\right),$$

where,

$$\hat{\delta} = \frac{\delta}{\delta + (1-\delta)\,q_k} \geqslant \delta, \tag{7}$$

[6] This is reminiscent of the neo-additive capacities introduced by Chateauneuf, Eichberger, and Grant (2007). Indeed if all the μ_k's are additive (that is, are conditional probabilities, and so, $\bar{\mu}_k = \mu_k$, for all k) then using expression (4) we see that such an PAJP capacity comes from the class of neo-additive capacities in which for all non-empty events $D \subsetneqq S$, $\nu(D) = (1-\delta)\,p(D) + \delta(1-\alpha)$, where p is an unconditional probability given by $p(D) = \sum_{k=1}^{K} q_k\mu_k(D \cap E_k)$.

with the inequality strict whenever $q_k < 1$. The convex component of the updated JP-capacity is given by

$$\mu'_k(A) = \frac{(1-\delta)\, q_k \mu_k(A \cap E_k)}{\delta + (1-\delta)\, q_k}. \tag{8}$$

The following result shows that a necessary and sufficient condition for the GBU updates on a partition to have the same ambiguity-attitude as the prior belief, is that the capacity be a PAJP-capacity.

Proposition 4.2 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a JP-capacity where μ is a* belief function *on S and $0 \leqslant \alpha \leqslant 1$. Assume that $|E_k| \geqslant 3$, for $1 \leqslant k \leqslant K$. Let ν_{E_k} denote the GBU update of ν conditional on E_k. Then a necessary and sufficient condition for ν_{E_k} to be a JP-capacity with the same α for $1 \leqslant k \leqslant K$ is that ν be a PAJP capacity, i.e. there exists a belief function μ_k on E_k, an additive probability distribution q on $\{1, ..., K\}$ and a number $\delta, 0 \leqslant \delta \leqslant 1$ such that for $A \subsetneqq S$:*

$$\mu(A) = (1-\delta) \sum_{k=1}^{K} q_k \mu_k(A \cap E_k), \qquad \mu(S) = 1.$$

The following example shows that it is not possible to drop the assumption $|E_k| \geqslant 3$, in Proposition 4.2.

Example 4.1 *Suppose that there are 4 states $S = \{s_1, s_2, s_3, s_4\}$. Let the partition be $E_1 = \{s_1, s_2\}, E_2 = \{s_3, s_4\}$. Let μ be the capacity whose Möbius inverse is $\beta_{s_i} = \eta$, for $1 \leqslant i \leqslant 4$; $\beta_{s_1 s_3} = \beta_{s_2 s_3} = \beta_{s_1 s_4} = \beta_{s_2 s_4} = \epsilon$; $\beta_S = 1 - 4\eta - 4\epsilon$; $\beta_E = 0$ for all other events E, where $\eta < \frac{1}{4}$ and $\epsilon < \frac{1}{4} - \eta$. Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$. We claim that the GBU updates of ν on E_1 and E_2 are JP capacities with the same α. This is proved in the appendix.*

4.3 Ex-Ante and Ex-Post Ambiguity

This section investigates how updating affects perceived ambiguity. Recall that a PAJP capacity is a convex combination of a Hurwicz capacity representing ambiguity about the states and K convex capacities which represent ambiguous beliefs about the signals. Before updating, the degree of ambiguity is a similar convex combination of the degree of ambiguity of the Hurwicz capacity, 1, receiving weight δ and K degrees of ambiguity of the signals $\lambda(\mu_j)$, receiving weight $(1-\delta)\, q_j$. Now suppose signal E_k is observed. Ex-post $K - 1$ of the signals are no longer possible. Thus the updated beliefs are represented by a capacity which is a convex combination of the Hurwicz capacity and the one signal capacity which is realized. Correspondingly the ex-post degree of ambiguity is a convex combination of the degree of ambiguity of the Hurwicz capacity (i.e. 1) and that of the signal actually observed, $\lambda(\mu_k)$. The following result finds expressions for ex-ante and ex-post ambiguity.

Proposition 4.3 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a PAJP capacity, where*

$$\mu(A) = (1-\delta)\sum_{j=1}^{K} q_j \mu_j (A \cap E_k) \, for A \subsetneqq S.$$

1. *The* ex-ante *degree of ambiguity of ν is $\lambda(\mu) = \delta + (1-\delta)\sum_{j=1}^{K} q_j \lambda(\mu_j)$.*

2. *If event E_k is observed then the* ex-post *degree of ambiguity is,*

$$\lambda(\mu_k') = \frac{\delta}{\delta + (1-\delta) q_k} + \frac{(1-\delta) q_k}{\delta + (1-\delta) q_k} \lambda(\mu_k).$$

In the ex-ante measure of ambiguity, 1 received weight δ and $\lambda(\mu_k)$ gets weight $(1-\delta) q_k$. The weights have been renormalized to ensure that they sum to unity. Ex post the weight on the Hurwicz capacity is greater. Thus for ex-post ambiguity to be lower it is necessary for the second term to be smaller to off-set this effect.

4.3.1 Comparative Statics

The comparative statics of updating are intuitive. Consider equation (2). As one would expect ex-post ambiguity is increasing in the ambiguity of the observed signal, i.e. the greater is $\lambda(\mu_k)$, the higher is ex-post ambiguity.

Ex post, the ambiguity is a convex combination of 1 and $\lambda(\mu_k)$. The weight on 1 is $\frac{\delta}{\delta+(1-\delta)q_k}$ while that on $\lambda(\mu_k)$ is $\frac{(1-\delta)q_k}{\delta+(1-\delta)q_k}$. Note that $1 \geqslant \lambda(\mu_k)$. Increasing δ (resp. decreasing q_k) decreases the weight on $\lambda(\mu_k)$ and increases the weight on 1 in the convex combination. Thus ex-post ambiguity is increasing in δ and decreasing in q_k. This is intuitive. The higher is δ the more ex-ante ambiguity there is over the states. As one would expect this increases ex-post ambiguity. The smaller is q_k the more unlikely is the signal. Thus seeing an unlikely signal increases ambiguity. The result below proves this formally.

Proposition 4.4 *Ex post ambiguity* $\lambda(\mu'_k)$ *is increasing in* δ *and decreasing in* q_k.

4.3.2 Examples

We now wish to investigate the factors which determine whether ambiguity increases or decreases after updating. For illustrative purposes we shall consider some special cases.

Case 1 The prior over the state space is unambiguous. This implies that the only source of ambiguity is the signals, i.e. $\delta = 0$. Then $\lambda(\mu) = \sum_{j=1}^{K} q_j \lambda(\mu_j)$ and $\lambda(\mu'_k) = \lambda(\mu_k)$. The degree of ambiguity increases/decreases as $\lambda(\mu_k) \gtrless \sum_{j=1}^{K} q_j \lambda(\mu_j)$. If the observed signal is less ambiguous than the average signal, then ambiguity will decrease after updating.

When the only source of ambiguity is from the signals, observing one of the less ambiguous signals will reduce ambiguity. This is intuitive since there is no longer the possibility of being exposed to the more ambiguous signals. By continuity, updating will have similar properties when δ is small, which implies that there is little ambiguity about the prior over the state space.

Case 2 The observed signal is unambiguous, $\lambda(\mu_k) = 0$. Ex-ante ambiguity is given by, $\lambda(\mu) = \delta + (1-\delta)\sum_{j\neq k} q_j \lambda(\mu_j)$. Ex-post ambiguity is given by, $\lambda(\mu'_k) = \frac{\delta}{\delta+(1-\delta)q_k}$. For ambiguity to be lower ex-post we require $\lambda(\mu) - \lambda(\mu'_k) \geqslant 0$.
Now $\lambda(\mu) - \lambda(\mu'_k) = \delta + (1-\delta)\sum_{j\neq k} q_j \lambda(\mu_j) - \frac{\delta}{\delta+(1-\delta)q_k} = (1-\delta)\frac{\delta q_k - \delta + (\delta+(1-\delta)q_k)\sum_{j\neq k} q_j \lambda(\mu_j)}{\delta+(1-\delta)q_k}$.

Hence

$$\lambda(\mu) - \lambda(\mu'_k) = (1-\delta)\frac{(\delta+(1-\delta)q_k)\sum_{j\neq k} q_j \lambda(\mu_j) - \delta(1-q_k)}{\delta+(1-\delta)q_k}. \tag{9}$$

Thus $\lambda(\mu) \gtrless \lambda(\mu'_k)$ as $\frac{1}{(1-q_k)}\left(\sum_{j\neq k} q_j \lambda(\mu_j)\right) \gtrless \frac{\delta}{(\delta+(1-\delta)q_k)}$. The lhs of this inequality is the average ambiguity of the signals ex-ante and the rhs is what the ambiguity of the states ex post would be if none of the signals were ambiguous.

An interesting sub-case is where the observed signal is unambiguous, while all the other signals display the maximal degree of ambiguity, i.e. $\lambda(\mu_k) = 0, \lambda(\mu_j) = 1$, $j \neq k$. These assumptions imply that $\left(\sum_{j\neq k} q_j \lambda(\mu_j)\right) = (1-q_k)$. Hence from equation (9), $\lambda(\mu) - \lambda(\mu'_k) = (1-\delta)(1-q_k)\frac{(\delta+(1-\delta)q_k)-\delta}{\delta+(1-\delta)q_k} = \frac{(1-\delta)^2 q_k(1-q_k)}{\delta+(1-\delta)q_k} > 0$. Thus in this extreme case, observing the least ambiguous signal will always decrease ambiguity. By continuity, if an individual observes a signal with much lower ambiguity than the other possible signals then ambiguity will be reduced.

Case 3 **All signals are equally ambiguous** This assumption implies, $\lambda\left(\mu_j\right) = \lambda, 1 \leqslant j \leqslant K$ for some $\lambda, 0 \leqslant \lambda \leqslant 1$. One may see that ambiguity always rises in this case. Ex-ante ambiguity is given by $\lambda\left(\mu\right) = \delta + (1-\delta)\lambda$, while ex-post ambiguity is given by $\lambda\left(\mu_k'\right) = \frac{\delta}{\delta+(1-\delta)q_k} + \frac{(1-\delta)q_k}{\delta+(1-\delta)q_k}\lambda$. Both are convex combinations of 1 and λ. Since $\frac{1}{\delta+(1-\delta)q_k} > 1$, the weight on 1 has increased in the expression for ex-post ambiguity. Thus provided $\delta > 0$, ex-post ambiguity is always larger than ex-ante ambiguity when the signals are equally ambiguous.

To summarise, ambiguity is more likely to be lower after updating:

1. the smaller is the ambiguity of the states i.e. δ;
2. if the observed signal is less ambiguous than average;
3. the more likely the observed signal is, (i.e. the higher is q_k).

4.4 Dynamic Consistency

In previous sections we have explored the implications of keeping ambiguity-attitude the same before and after updating. This can be viewed as a weak form of dynamic consistency. The condition is clearly necessary but not sufficient for dynamic consistency. Here we explore the relation between this condition and full dynamic consistency. We find that the necessary and sufficient conditions for dynamic consistency are only slightly stronger. This is intuitive since changes in ambiguity-attitude when updating are likely to be a major cause of dynamic inconsistency. First we shall define dynamic consistency

Definition 4.2 *Preferences are said to be* dynamically consistent *with respect to a partition,* $E_1, ..., E_K$ *if* $\int u(a)\, d\nu \geqslant \int u(b)\, d\nu$ *implies* $\int u(a_k)\, d\nu_{E_k} \geqslant \int u(b_k)\, d\nu_{E_k}$, *for* $1 \leqslant k \leqslant K$. *Here* $a_k \in A(E_k)$ *denotes the restriction of act* a *to the event* E_k, *where* $A(E_k)$ *denotes the set of all acts on* E_k, *i.e. the set of all functions* $f : E_k \to X$.

This implies that if act a is chosen in the first period the decision-maker will not wish to revise his/her decision in the second period no matter which element of the partition is observed.

First we make some additional assumptions. In particular, we shall assume that the utility function is continuous and that no state is null in the sense that increasing the utility in that state will always lead to a strictly preferred option.

Assumption 4.1 *The utility function* $u : X \to \mathbb{R}$ *is continuous.*

Assumption 4.2 (Strong Monotonicity) *For two acts* $a, b \in A(S)$, *if* $\exists \hat{s} \in S$, *such that* $u(a(\hat{s})) > u(b(\hat{s}))$ *and* $\forall s \in S, u(a(s)) \geqslant u(b(s))$ *then* $a \succ b$.[7]

Definition 4.3 *We say that the partition* $E_1, ..., E_K$ *is* non-trivial, if $K \geqslant 2$ *and* $|E_k| \geqslant 2$, *for* $1 \leqslant k \leqslant K$.

The next result finds necessary and sufficient conditions for dynamic consistency. The condition is that the convex part of the JP-capacity be additive over the given partition.

Proposition 4.5 *Let* $E_1, ..., E_K$ *be a non-trivial partition of* S. *If a decision-maker has CEU preferences, which satisfy Assumptions 4.1 and 4.2 with beliefs represented by a JP-capacity* $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, *where* $\alpha \neq \frac{1}{2}$, *and (s)he updates his/her preferences with GBU updating then the following conditions are equivalent:*

1. *(s)he is dynamically consistent,*

2. $\sum_{k=1}^{K} \mu(E_k) = 1$.

As far as we are aware this is the first result on the dynamic consistency of JP capacities. Both the result and the proof are extensions of Theorem 2.1 in Eichberger, Grant, and Kelsey (2005) who, in turn, extended an earlier result in Sarin and Wakker (1998). The main difference is that we have dropped the assumption of ambiguity-aversion made in the earlier paper. In the case where $\alpha = \frac{1}{2}$ the sufficiency proof still holds, however we conjecture that this condition is no longer necessary for dynamic consistency.

As already noted, keeping ambiguity-attitude unchanged when updating can be seen as a weak form of dynamic consistency. Comparing Propositions 4.2 and 4.5 we notice that the only additional restriction imposed by full dynamic consistency is that $\delta = 0$. This implies that first period beliefs are additive or there is no ambiguity about which signal we shall see. Thus

[7] We do not use the full strength of this assumption. In fact we only need it to apply to the events C and D in the proof of Proposition 4.5.

the necessary and sufficient conditions for keeping α constant are close to those for dynamic consistency. It is clear that changes in ambiguity-attitude could be a source of dynamic inconsistency. This result shows that when $\delta = 0$ changes in ambiguity-attitude are the *only* reason for violations of dynamic consistency.

5 CONCLUSION

This paper studies learning and ambiguity. We have extended previous work on updating ambiguous beliefs by allowing for the possibility of ambiguity-seeking behaviour in some choices and ambiguity avoiding in others. The main principle used in this paper is that ambiguity-attitude should be preserved by updating, while beliefs and perceptions of ambiguity may be revised when new information is received. We believe this principle may be applicable more generally.

One might note that there is a connection between the results of section 3 and 4. If we consider updates on an arbitrary event, the condition for dynamic consistency is that beliefs be additive and the condition for ambiguity-attitude to be constant is that beliefs be neo-additive. If we only consider updates on a given partition, the condition for dynamic consistency is that beliefs be additive over that partition and the condition for ambiguity-attitude to be constant is that they be neo-additive over the partition.[8]

This paper has looked for classes of preferences which are closed under GBU updating. In other words we require the updated preferences to have the same functional form and the same ambiguity-attitude as the original preference. It is desirable that a class of preferences be closed under updating, since in practice the prior will itself be an update of an earlier belief.[9] If we require the updates on all events to lie in the same class as the original preferences, then, under mild assumptions, the preferences must be neo-additive.

A PROOFS

This appendix contains proofs of those results not proved in the text.

[8] By neo-additive we mean additive except on events where extreme outcomes occur.

[9] This is reminiscent of the notion from statistics of a conjugate family of probability distributions for which a posterior distribution has the same form as the prior from which it was updated.

Proof of Lemma 3.1 Consider $A \subsetneqq E$, then

$\nu_E(A) = \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))+1-\alpha\mu(A\cup E^c)-(1-\alpha)(1-\mu((A\cup E^c)^c))}$

$= \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha[\mu(A)-1+\mu(B\cup E^c)-\mu(A\cup E^c)+(1-\mu(B))]+1-\mu(B\cup E^c)+\mu(B)}$

$= \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(B\cup E^c))}{\alpha[\mu(A)-\mu(A\cup E^c)+\mu(B\cup E^c)-\mu(B)]+1-\mu(B\cup E^c)+\mu(B)}.$

Sufficiency If $\mu(A\cup E^c) - \mu(A) = \mu(B\cup E^c) - \mu(B)$, then

$$\nu_E(A) = \frac{\alpha\mu(A)}{1-\mu(B\cup E^c)+\mu(B)} + (1-\alpha)\left(\frac{1-\mu(B\cup E^c)}{1-\mu(B\cup E^c)+\mu(B)}\right) \tag{10}$$

$= \frac{\alpha\mu(A)}{1-\mu(A\cup E^c)+\mu(A)} + (1-\alpha)\left(\frac{1-\mu(A\cup E^c)+\mu(A)-\mu(B)}{1-\mu(A\cup E^c)+\mu(A)}\right)$

$$= \frac{\alpha\mu(A)}{1-\mu(A\cup E^c)+\mu(A)} + (1-\alpha)\left(1 - \frac{\mu(E\backslash A)}{1-\mu(A\cup E^c)+\mu(A)}\right),$$

which has the JP form with the same ambiguity-attitude parameter α.

Necessity If $\nu_E^\alpha(A) = \alpha\sigma(A) + (1-\alpha)\bar{\sigma}(A)$, where $0 \leqslant \alpha \leqslant 1$ and σ is a convex capacity on E, then $\frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha[\mu(A)-\mu(A\cup E^c)+\mu(B\cup E^c)-\mu(B)]+1-\mu(B\cup E^c)+\mu(B)} = \alpha\sigma(A) + (1-\alpha)\bar{\sigma}(A)$.

This equation has the form $\frac{a\alpha+b}{c\alpha+d} = e\alpha + f$, where $c = \mu(A) + \mu(A^c) - \mu(A\cup E^c) - \mu((A\cup E^c)^c)$, etc. Cross multiplying, $a\alpha + b = \alpha^2 ce + (fc+de)\alpha + fd$. Equating coefficients we obtain: $ce = 0, a = (fc+de), b = fd$.

Unless σ is the complete uncertainty capacity, there exists A such that $\sigma(A) = e \neq 0$, which implies $c = 0$. (Note one can easily show that the result holds if σ is the complete uncertainty capacity.) Hence $\mu(A) - \mu(A\cup E^c) + \mu(B\cup E^c) - \mu(B)$, holds. ■

Proposition 3.1 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a JP-capacity where μ is a* belief function *on S and $0 \leqslant \alpha \leqslant 1$ and $|S| \geqslant 4$. Let ν_E denote the GBU update of ν conditional on E. Then a necessary and sufficient condition for ν_E to be a JP-capacity with the same α for all $E \subsetneqq S$ is that ν be neo-additive.*

Proof. Sufficiency of this condition follows from Proposition 1 of Eichberger, Grant, and Kelsey (2010).

Necessity Suppose that μ is a belief function and let β denote the Möbius inverse of μ. It is sufficient to show $\beta_B = 0$ unless $B = S$ or B is a singleton.

Let $\hat{s}$ denote a given state. Let $E = S \backslash \hat{s}$, then $E^c = \{\hat{s}\}$. Take $\sigma \in E$. Let $A = E \backslash \{\sigma\}$ and $B = \{\sigma\}$. Then by equation (2), $\mu(A \cup E^c) - \mu(A) = \mu(B \cup E^c) - \mu(B)$. Rewriting this in terms of the Möbius inverse we obtain: $\sum_{D \subseteq A \cup E^c} \beta_D - \sum_{D \subseteq A} \beta_D = \sum_{D \subseteq \sigma \cup E^c} \beta_D - \beta_\sigma$. This may be reorganized as,

$\sum_{D \subseteq A} \beta_D + \beta_{\hat{s}} + \sum_{D \subseteq A} \beta_{D \cup \hat{s}} - \sum_{D \subseteq A} \beta_D = \beta_\sigma + \beta_{\hat{s}} + \beta_{\sigma\hat{s}} - \beta_\sigma$.

Simplifying

$$\beta_{\sigma\hat{s}} = \sum_{D \subseteq A} \beta_{D \cup \hat{s}}. \tag{11}$$

Hence $\beta_{\sigma\hat{s}} \geqslant \sum_{s' \neq \hat{s}, \sigma} \beta_{s'\hat{s}}$, since we have deleted some non-negative terms from the rhs.

Summing over σ, $\sum_{\sigma \neq \hat{s}} \beta_{\sigma\hat{s}} \geqslant \sum_{\sigma \neq \hat{s}} \sum_{s' \neq \hat{s}, \sigma} \beta_{s'\hat{s}} = (n-2) \sum_{s' \neq \hat{s}} \beta_{s'\hat{s}}$.[10] Note that the two sums are identical. Hence if $n \geqslant 4$ this implies $\beta_{s'\hat{s}} = 0$ for all $s', \hat{s} \in S$. Substituting into equation (11), $\sum_{D \subseteq A} \beta_{D \cup \hat{s}} = 0$. Since $\beta_{D \cup \hat{s}} \geqslant 0$, this implies $\beta_{D \cup \hat{s}} = 0$ for all $D \subseteq A$. Bearing in mind that $\hat{s}$ and A were chosen arbitrarily this establishes that $\beta_G = 0$, for all $G, 2 \leqslant |G| \leqslant n-1$. ■

Proposition A.1 *If we define a JP-capacity ν, by $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where μ is the capacity from Example 3.1 then for all $E \subseteq S$:*

1. *The GBU update ν_E is a JP-capacity with the same ambiguity-attitude parameter α. However ν is not neo-additive and μ is not a belief function;*

2. *If E is any 3-element event. Then the GBU update of ν conditional on E is $\nu_E = \alpha\mu_E + (1-\alpha)\bar{\mu}_E$, where μ_E is the symmetric convex capacity on E defined by $\mu_E(0) = 0, \mu_E(1) = \frac{\eta}{1-\eta-\epsilon}, \mu_E(2) = \frac{2\eta+\epsilon}{1-\eta-\epsilon}$ and $\mu_E(3) = 1$;*

3. *If E is any 2-element event $\nu_E = \alpha\mu_E + (1-\alpha)\bar{\mu}_E$ where μ_E is the symmetric convex capacity on E given by $\mu_E(0) = 0, \mu_E(1) = \frac{\eta}{1-2\eta-2\epsilon}, \mu_E(2) = 1$.*

Proof. First we shall show that μ is convex and satisfies equation (2), which establishes that the GBU update of ν is a JP capacity with the same α by Lemma 3.1. Equation (2) requires that $\mu(3) - \mu(2) = \mu(2) - \mu(1)$ or $3\eta + 2\epsilon - (2\eta + \epsilon) = 2\eta + \epsilon - \eta$, which clearly holds.

Convexity is satisfied since:

[10] Since E contains $n-2$ elements other than s'.

1. $1 \geqslant 2\mu(3) - \mu(2) \Leftrightarrow 1 \geqslant 6\eta + 4\epsilon - (2\eta + \epsilon) = 4\eta + 2\epsilon$, which holds since $\eta < \frac{1}{4}$ and $\epsilon < \frac{1}{4} - \eta$;
2. $\mu(3) \geqslant 2\mu(2) - \mu(1) \Leftrightarrow 3\eta + 2\epsilon \geqslant 2(2\eta + \epsilon) - \eta$, which always holds;
3. $\mu(2) \geqslant 2\mu(1) \Leftrightarrow \mu(2) = 2\eta + \epsilon \geqslant 2\eta$.

The Möbius inverse of μ is: $\beta_1 = \eta, \beta_2 = \epsilon, \beta_3 = -\epsilon$, where β_j denotes the Möbius inverse of a set with j states for $1 \leqslant j \leqslant 3$. Since the Möbius inverse has some negative values, μ is not a belief function.

To show the updates have the given form We only need to consider the updates conditional on 2 and 3-element events, since updating on a 1-element event is trivial. Let E be an arbitrary 3-element event. Let C be a 2-element subset of E. By equation (10) the GBU-update is given by: $\nu_E(C) = \frac{\alpha\mu(C)+(1-\alpha)(1-\mu(C^c))}{1-\mu(C^c)+\mu((C\cup E^c)^c)} = \frac{\alpha(2\eta+\epsilon)+(1-\alpha)(1-(2\eta+\epsilon))}{1-(2\eta+\epsilon)+\eta}$. Thus

$$\nu_E(C) = \frac{\alpha(2\eta+\epsilon)}{1-\eta-\epsilon} + (1-\alpha)\left(1 - \frac{\eta}{1-\eta-\epsilon}\right). \tag{12}$$

Similarly if G is a 1-element subset of E. By equation (10) the GBU-update is

$$\nu_E(G) = \alpha\frac{\eta}{1-\eta-\epsilon} + (1-\alpha)\left(1 - \frac{2\eta+\epsilon}{1-\eta-\epsilon}\right). \tag{13}$$

This establishes part (2).

Now consider the updates of ν conditional on a 2-element event. Let E denote an arbitrary 2-element event, let A be a non-trivial subset of E and let $B = E \backslash A$. Then by equation (10) the GBU update is given by $\nu_E(A) = \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{1-\mu(A^c)+\mu((A\cup E^c)^c)} = \frac{\alpha\eta+(1-\alpha)(1-(3\eta+2\epsilon))}{1-(3\eta+2\epsilon)+\eta}$

$$= \alpha\left(\frac{\eta}{1-2\eta-2\epsilon}\right) + (1-\alpha)\left(1 - \frac{\eta}{1-2\eta-2\epsilon}\right).$$

Thus ν_E has the JP form with the same ambiguity-attitude parameter and updated convex capacity μ_E given by $\mu_E(0) = 0, \mu_E(1) = \frac{\eta}{1-2\eta-2\epsilon}, \mu_E(2) = 1$. Note that $1 - 2\eta - 2\epsilon \geqslant 1 - 2\eta - \frac{1}{2} + 2\eta = \frac{1}{2}$. Since $\eta < \frac{1}{4}$ this implies that μ_E is convex. ■

The example may be understood by considering the symmetric neo-additive capacity defined by $\kappa(A) = |A|(\eta + \epsilon), A \subsetneqq S, \kappa(S) = 1$. Let a be an act such that $a(s_1) > ... > a(s_4)$. Then the Choquet integral of a with respect to κ is:

$$a(s_1)(\eta+\epsilon)+a(s_2)(\eta+\epsilon)+a(s_3)(\eta+\epsilon)+a(s_4)(1-3\eta-3\epsilon).$$

Compare this with the Choquet integral of a with respect to μ:

$$a(s_1)(\eta)+a(s_2)(\eta+\epsilon)+a(s_3)(\eta+\epsilon)+a(s_4)(1-3\eta-2\epsilon).$$

One can see that μ is similar to κ except that it under-weights the best outcome as well as over-weighting the worst outcome in the Choquet integral.

Proof of Proposition 3.2 It is clear that if ν is additive, the DS update has the required property. Thus sufficiency is immediate.

Necessity Suppose that $\hat{\nu}_E^{\alpha}(A)=\alpha\sigma(A)+(1-\alpha)\bar{\sigma}(A)$, where $0\leqslant\alpha\leqslant 1$ and σ is a convex capacity on E.[11] Let $B=E\backslash A$.

Then $\frac{\alpha\mu(A\cup E^c)+(1-\alpha)(1-\mu((A\cup E^c)^c))-\alpha\mu(E^c)-(1-\alpha)(1-\mu(E))}{1-\alpha\mu(E^c)-(1-\alpha)(1-\mu(E))}=\alpha\sigma(A)+(1-\alpha)\bar{\sigma}(A)$. The lhs may be rearranged as: $\frac{\alpha[1-\mu(E)+\mu(A\cup E^c)-\mu(E^c)]+(1-\mu(B))-\alpha(1-\mu(B))-1+\mu(E)}{\mu(E)+\alpha(1-\mu(E)-\mu(E^c))}$.

Thus $\frac{\alpha[\mu(B)+\mu(A\cup E^c)-\mu(E^c)-\mu(E)]+\mu(E)-\mu(B)}{\mu(E)+\alpha[1-\mu(E)-\mu(E^c)]}=\alpha\sigma(A)+(1-\alpha)\bar{\sigma}(A)$. This equation has the form $\frac{a\alpha+b}{c\alpha+d}=e\alpha+f$, where $c=1-\mu(E)-\mu(E^c)$ etc. Cross multiplying, $a\alpha+b=\alpha^2ce+(fc+de)\alpha+fd$. Equating coefficients we obtain: $ce=0, a=(fc+de), b=fd$.

Unless σ is the complete uncertainty capacity, there exists A such that $\sigma(A)=e\neq 0$, which implies $c=0$. (Note one can easily show that the result holds if σ is the complete uncertainty capacity.) Since μ is convex, $\mu(E)+\mu(E^c)=1$ for all $E\subseteq S$, which implies that μ is additive. ■

Proof of Proposition 3.3 It is clear that if ν is additive the Optimistic update has the required property.

Necessity If the update is a JP-capacity with the same α, we have

[11] In this proof, ν_E and ν_E^{α} denote the Dempster-Shafer update of ν conditional on event E.

$$\bar{\nu}_E^{\alpha}(A) = \frac{\alpha\mu(A)+(1-\alpha)(1-\mu(A^c))}{\alpha\mu(E)+(1-\alpha)(1-\mu(E^c))}$$

$$= \frac{\alpha\mu(A) + (1-\alpha)(1-\mu(A^c))}{1-\mu(E^c) + \alpha[\mu(E) + \mu(E^c) - 1]} = \alpha\sigma(A) + (1-\alpha)\bar{\sigma}(A),$$

where $0 \leqslant \alpha \leqslant 1$ and σ is a convex capacity on E.[12] As in the proof of Proposition 3.2, this implies that the term multiplying α in the denominator must be 0, i.e. $\mu(E) + \mu(E^c) = 1$. Since μ is convex, $\mu(E) + \mu(E^c) = 1$ for all $E \subseteq S$, implies that μ is additive. ■

Proposition 4.1 *The GBU update of the PAJP-capacity, ν conditional on event E_k is given by: $\hat{\nu}_k(A) = \alpha\left(\frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right) + (1-\alpha)\left(1 - \frac{(1-\delta)q_k\mu_k(A^c\cap E_k)}{\delta+(1-\delta)q_k}\right)$, where, $\hat{\delta} = \frac{\delta}{\delta+(1-\delta)q_k} \geqslant \delta$, with the inequality strict whenever $q_k < 1$. The convex component of the updated JP-capacity is given by $\hat{\mu}_k(A) = \frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}$.*

Proof. Suppose that E_k is observed. Let $\hat{\nu}_k$ denote the GBU update of ν conditional on E_k. By definition, $\hat{\nu}_k(A) = \frac{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)}{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)+\alpha\delta+(1-\delta)q_k\bar{\nu}_k(A^c\cap E_k)}$

$$= \frac{(1-\alpha)\delta + (1-\delta)q_k\nu_k(A\cap E_k)}{\delta + (1-\delta)q_k}.$$

Set $\hat{\delta} := \frac{\delta}{\delta+(1-\delta)q_k}$ and we obtain the right-hand side expression of equation (7). To obtain the expression in equation (8), notice that $\frac{(1-\alpha)\delta+(1-\delta)q_k\nu_k(A\cap E_k)}{\delta+(1-\delta)q_k}$

$$\begin{aligned}
&= \frac{(1-\alpha)\delta + (1-\delta)q_k[\alpha\mu_k(A\cap E_k) + (1-\alpha)\bar{\mu}_k(A\cap E_k)]}{\delta+(1-\delta)q_k} \\
&= \alpha\left(\frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right) + (1-\alpha)\left(\frac{\delta + (1-\delta)q_k\bar{\mu}_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right)
\end{aligned}$$

$$= \alpha\left(\frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}\right) + (1-\alpha)\left(1 - \frac{(1-\delta)q_k\mu_k(A^c\cap E_k)}{\delta+(1-\delta)q_k}\right).$$

■

The following lemma assumes that ambiguity-attitude is constant and shows that if a set consists of the union of subsets of two different elements of the partition then its Möbius inverse must be zero.

Lemma A.1 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be a JP-capacity where μ is a* belief function *on S and $0 \leqslant \alpha \leqslant 1$. Assume that $|E_k| \geqslant 3$, for $1 \leqslant k \leqslant K$. Let ν_{E_k} denote the GBU update of ν*

[12] In this proof, ν_E and ν_E^{α} denote the Optimistic update of ν conditional on event E.

conditional on E_k. Then a necessary and sufficient condition for ν_{E_k} to be a JP-capacity with the same α for $1 \leqslant k \leqslant K$ is that for $A \subseteq E_k$, and for all non-empty $F \subseteq E_k^c, \beta_{A \cup F} = 0$, for $1 \leqslant k \leqslant K$.

Proof. Sufficiency follows from Proposition 4.1.

Necessity Let μ be a belief function and let β denote the Möbius inverse of μ. By equation (2) for all A, B such that $A \cup B = E_k, A \cap B = \varnothing$.

$$\mu\left(A \cup E_k^c\right) - \mu\left(A\right) = \mu\left(B \cup E_k^c\right) - \mu\left(B\right). \tag{14}$$

Consider a given element of the partition E_k. Assume $E_k = \{\sigma_1, ..., \sigma_L\}$. We claim that for $A \subseteq E_k$, all non-empty $F \subseteq E_k^c, \beta_{A \cup F} = 0$. We shall proceed by induction on the number of states in A.

Step 1 $|A| = 1$. In this case $A = \{\sigma_\ell\}$ for some $\ell, 1 \leqslant \ell \leqslant L$. By equation (14)

$\mu\left(\sigma_\ell \cup E_k^c\right) - \mu\left(\sigma_\ell\right) = \mu\left(\left(E_k \backslash \sigma_\ell\right) \cup E_k^c\right) - \mu\left(E_k \backslash \sigma_\ell\right)$. Rewriting in terms of the Möbius inverse, $\sum_{D \subseteq \left(\sigma_\ell \cup E_k^c\right)} \beta_D - \beta_{\{\sigma_\ell\}} = \sum_{D \subseteq (E_k \backslash \sigma_\ell) \cup E_k^c} \beta_D - \sum_{D \subseteq (E_k \backslash \sigma_\ell)} \beta_D$

or $\sum_{D \subseteq E_k^c} \beta_D + \sum_{D \subseteq E_k^c} \beta_{D \cup \sigma_\ell} - \beta_{\{\sigma_\ell\}} = \sum_{D \subseteq E_k^c} \beta_D + \sum_{\substack{D \subseteq (E_k \backslash \sigma_\ell) \cup E_k^c \\ D \nsubseteq E_k^c}} \beta_D - \sum_{D \subseteq (E_k \backslash \sigma_\ell)} \beta_D$,

which implies $\sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} = \sum_{D \subseteq (E_k \backslash \sigma_\ell) \cup E_k^c, D \nsubseteq E_k^c, D \nsubseteq (E_k \backslash \sigma_\ell)} \beta_D$.

Hence $\sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} \geqslant \sum_{j \neq \ell} \sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_j}$, since we have deleted some non-negative terms from the rhs. Summing over ℓ, $\sum_{\ell=1}^{L} \sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} \geqslant (L-1) \sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell}$. Since $L \geqslant 3$, this implies $\sum_{\substack{D \subseteq E_k^c \\ D \neq \varnothing}} \beta_{D \cup \sigma_\ell} = 0$. Since $\beta_{D \cup \sigma_\ell} \geqslant 0$, for $1 \leqslant \ell \leqslant L$, we may deduce $\beta_{D \cup \sigma_\ell} = 0$, for all non-empty $D \subseteq E_k^c$. This establishes the result in the case where $|A| = 1$.

Inductive step Now take a given set $A \subseteq E_k$. Our inductive hypothesis is that for all strictly smaller subsets B of E_k, $\beta_{B \cup F} = 0$, for all non-empty $F \subseteq E_k^c$. There are two cases to consider.

Case 1 $|A| \leqslant \frac{L}{2}$ In this case we may choose $G \subseteq E_k$ such that $|G| = |A| - 1$ and $G \cap A = \varnothing$. Let $H = E_k \backslash G$. Note that $A \subseteq H$. By equation (2) $\mu\left(G \cup E_k^c\right) - \mu\left(G\right) = \mu\left(H \cup E_k^c\right) - \mu\left(H\right)$. Rewriting this in terms of the Möbius inverse we obtain:

$\sum_{D \subseteq \left(G \cup E_k^c\right)} \beta_D - \sum_{D \subseteq G} \beta_D = \sum_{D \subseteq \left(H \cup E_k^c\right)} \beta_D - \sum_{D \subseteq H} \beta_D$.

Expanding $\sum_{D\subseteq E_k^c}\beta_D+\sum_{D\subseteq G}\beta_D+\sum_{D\subseteq\left(G\cup E_k^c\right),D\nsubseteq G,D\nsubseteq E_k^c}\beta_D-\sum_{D\subseteq G}\beta_D$

$=\sum_{D\subseteq E_k^c}\beta_D+\sum_{D\subseteq H}\beta_D+\sum_{D\subseteq\left(H\cup E_k^c\right),D\nsubseteq H,D\nsubseteq E_k^c}-\sum_{D\subseteq H}\beta_D.$

This may be simplified to:

$$\sum_{D\subseteq\left(G\cup E_k^c\right),D\nsubseteq G,D\nsubseteq E_k^c}\beta_D=\sum_{D\subseteq\left(H\cup E_k^c\right),D\nsubseteq H,D\nsubseteq E_k^c}\beta_D. \tag{15}$$

Recall that by the inductive hypothesis $\beta_{B\cup F}=0$, for subsets B of E_k strictly smaller than A and non-empty $F\subseteq E_k^c$. Thus all terms on the lhs of equation (15) are zero. i.e. $0=\sum_{D\subseteq\left(H\cup E_k^c\right),D\nsubseteq H,D\nsubseteq E_k^c}\beta_D$. Since μ is, by assumption, a belief function, all the β's are non-negative, which implies $\beta_D=0$ for all $D\subseteq\left(H\cup E_k^c\right),D\nsubseteq E_k^c D\nsubseteq H$. In particular $\beta_{A\cup F}=0$, for all non-empty $F\subseteq E_k^c$. This completes the proof of this case.

Case 2, $|A|>\frac{L}{2}$ Let $Q=E_k\backslash A$. Then $|A|>|Q|$. By equation (2) $\mu\left(A\cup E_k^c\right)-\mu(A)=\mu\left(Q\cup E_k^c\right)-\mu(Q)$. Rewriting this in terms of the Möbius inverse we obtain:

$\sum_{D\subseteq\left(A\cup E_k^c\right)}\beta_D-\sum_{D\subseteq A}\beta_D=\sum_{D\subseteq\left(Q\cup E_k^c\right)}\beta_D-\sum_{D\subseteq Q}\beta_D.$

As in case 1 this may be simplified to:

$$\sum_{D\subseteq\left(A\cup E_k^c\right),D\nsubseteq A,D\nsubseteq E_k^c}\beta_D=\sum_{D\subseteq\left(Q\cup E_k^c\right),D\nsubseteq Q,D\nsubseteq E_k^c}\beta_D. \tag{16}$$

Recall that by the inductive hypothesis $\beta_{B\cup F}=0$, for subsets B of E_k strictly smaller than A and non-empty $F\subseteq E_k^c$. Thus all terms on the rhs of equation (16) are zero, hence $\sum_{D\subseteq\left(A\cup E_k^c\right),D\nsubseteq A,D\nsubseteq E_k^c}\beta_D=0$. As before, this implies $\beta_{A\cup F}=0$, for all non-empty $F\subseteq E_k^c$. This completes the proof of the inductive step. The result follows. ■

Proof of Proposition 4.2 **Sufficiency** Proposition 4.1 has already established sufficiency.

Necessity Now assume that ν_{E_k} is a JP-capacity with the same α for $1\leqslant k\leqslant K$ and μ is a belief function. Let $\delta=\beta_S\geqslant 0$. Then $\sum_{D\subsetneqq S}\beta_D=1-\delta$.

For $1\leqslant k\leqslant K$, define $q_k=\frac{1}{1-\delta}\sum_{D\subseteq E_k}\beta_B$. If $q_k\neq 0$ define a capacity μ_k on E_k by $\mu_k(A)=\frac{1}{(1-\delta)q_k}\sum_{D\subseteq A}\beta_B$ for $A\subseteq E_k$. It is clear that μ_k is convex since its Möbius inverse is non-negative. If $q_k=0$, define μ_k by $\mu_k(A)=0,A\subsetneqq E_k;\mu_k(E_k)=1$. If B is an arbitrary (proper) subset of S, then

$$\mu(B) = \sum_{D \subseteq B} \beta_D = \sum_{k=1}^{K} \sum_{D \subseteq B \cap E_k} \beta_D + \sum_{\substack{D \subseteq B \\ D \nsubseteq B \cap E_k}} \beta_D.$$

By Lemma A.1 of $A \subseteq E_k$, for all non-empty $F \subseteq E_k^c, \beta_{A \cup F} = 0$, for $1 \leqslant k \leqslant K$, hence the last sum is zero. Thus $\mu(B) = \sum_{k=1}^{K} \sum_{D \subseteq B \cap E_k} \beta_D = (1-\delta) \sum_{k=1}^{K} q_k \mu_k (B \cap E_k)$. Clearly $\mu(S) = 1$. Thus ν is a PAJP capacity. ■

Example 4.1 *Suppose that there are 4 states $S = \{s_1, s_2, s_3, s_4\}$. Let the partition be $E_1 = \{s_1, s_2\}, E_2 = \{s_3, s_4\}$. Let μ be the capacity whose Möbius inverse is $\beta_{s_i} = \eta$, for $1 \leqslant i \leqslant 4$; $\beta_{s_1 s_3} = \beta_{s_2 s_3} = \beta_{s_1 s_4} = \beta_{s_2 s_4} = \epsilon$; $\beta_S = 1 - 4\eta - 4\epsilon$; $\beta_E = 0$ for all other events E, where $\eta < \frac{1}{4}$ and $\epsilon < \frac{1}{4} - \eta$. Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$. We claim that the GBU updates of ν on E_1 and E_2 are JP capacities with the same α.*

Proof for Example 4.1 By definition

$\nu_{E_1}(s_1) = \frac{\alpha\mu(s_1)+(1-\alpha)(1-\mu(s_2,s_3,s_4))}{\alpha\mu(s_1)+(1-\alpha)(1-\mu(s_2,s_3,s_4))+1-\alpha\mu(s_1,s_3,s_4)-(1-\alpha)(1-\mu(s_2))}$

$= \frac{\alpha\eta+(1-\alpha)(1-(3\eta+2\epsilon))}{\alpha\eta+(1-\alpha)(1-(3\eta+2\epsilon))+1-\alpha(3\eta+2\epsilon)-(1-\alpha)(1-\eta)} = \frac{\alpha\eta+(1-\alpha)(1-(3\eta+2\epsilon))}{1-2\eta-2\epsilon}$

$= \alpha\frac{\eta}{1-2\eta-2\epsilon} + (1-\alpha)\left(1 - \frac{\eta}{1-2\eta-2\epsilon}\right).$

By symmetry $\nu_{E_1}(s_2) = \frac{\alpha\eta}{1-2\eta-\epsilon} + (1-\alpha)\left(1 - \frac{\eta}{1-2\eta-\epsilon}\right)$. Thus ν_{E_1} has the JP-form with the same α as ν. By symmetry ν_{E_2} also has the JP-form with the same α as ν. ■

Proposition 4.3 *Let $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$ be an PAJP capacity, where $\mu(A) = (1-\delta)\sum_{j=1}^{K} q_j \mu_j (A \cap E_k)$ for $A \subsetneqq S$.*

1. *The ex-ante degree of ambiguity of ν is $\lambda(\mu) = \delta + (1-\delta)\sum_{j=1}^{K} q_j \lambda(\mu_j)$.*

2. *If event E_k is observed then the ex-post degree of ambiguity is,*

$$\lambda(\mu'_k) = \frac{\delta}{\delta + (1-\delta) q_k} + \frac{(1-\delta) q_k}{\delta + (1-\delta) q_k}\lambda(\mu_k).$$

Proof. For $1 \leqslant k \leqslant K$, let $F_k \subseteq E_k$ be such that $\lambda(\mu_k) = \bar{\mu}_k(F_k) - \mu_k(F_k)$. Define $F = \bigcup_{k=1}^{K} F_k$. Now let A be an arbitrary subset of S. Then $\bar{\mu}(A) - \mu(A)$

$$= \delta + (1-\delta)\sum_{k=1}^{K} q_k [1 - \mu_k(A^c \cap E_k) - \mu_k(A \cap E_k)] \leqslant \delta + (1-\delta)\sum_{k=1}^{K} q_k \lambda(\mu_k), \tag{17}$$

which establishes that $\lambda(\mu) \geqslant \delta + (1-\delta)\sum_{k=1}^{K} q_k \lambda(\mu_k)$. Note also that equation (17) holds with equality if $A = F$, which implies $\lambda(\mu) \leqslant \delta + (1-\delta)\sum_{k=1}^{K} q_k \lambda(\mu_k)$.

From Proposition 4.1, if event E_k is observed, the updated capacity $\nu'(A) = \alpha\mu'_k(A) + (1-\alpha)\bar{\mu}'_k(A)$, where $\mu'_k(A) = \frac{(1-\delta)q_k\mu_k(A\cap E_k)}{\delta+(1-\delta)q_k}$. Thus the ex-post degree of ambiguity is,

$\lambda(\mu'_k) = \max_{A\subseteq E_k}\{\bar{\mu}'_k(A) - \mu'_k(A)\}$

$= \max_{A\subseteq E_k}\left\{1 - \frac{(1-\delta)q_k}{\delta+(1-\delta)q_k} + \frac{(1-\delta)q_k}{\delta+(1-\delta)q_k}[\bar{\mu}_k(A) - \mu_k(A)]\right\}$

$= \frac{\delta}{\delta+(1-\delta)q_k} + \frac{(1-\delta)q_k}{\delta+(1-\delta)q_k}\lambda(\mu_k)$. ■

Proof of Proposition 4.4 The effect of the likelihood of the signal on ex-post ambiguity can be measured by the derivative, $\frac{\partial\lambda(\mu'_k)}{\partial\delta} = \frac{\delta+q_k-\delta q_k-\delta+\delta q_k}{(\delta+(1-\delta)q_k)^2} - \left(\frac{\delta+q_k-\delta q_k+1-q_k-\delta_k+\delta q_k}{(\delta+(1-\delta)q_k)^2}\right) q_k\lambda(\mu_k)$ (by the quotient rule), $= q_k\frac{(1-\lambda(\mu_k))}{(\delta+(1-\delta)q_k)^2} > 0$.

Thus an increase in the ex-ante ambiguity over the state space increases ex-post ambiguity. Similarly, $\frac{\partial\lambda(\mu'_k)}{\partial q_k} = (1-\delta)\frac{(\delta+(1-\delta)q_k)\lambda(\mu_k)-\delta-(1-\delta)\lambda(\mu_k)}{(\delta+(1-\delta)q_k)^2}$

$= -(1-\delta)\frac{\delta(1-\lambda(\mu_k))+(1-\delta)(1-q_k)\lambda(\mu_k)}{(\delta+(1-\delta)q_k)^2} < 0$. Thus an increase in the likelihood of the signal, q_k, decreases ex-post ambiguity. ■

Lemma A.2 *Let $E_1, ..., E_K$ be a partition and let σ be a convex or concave capacity on S such that $\sum_{i=1}^{K}\sigma(E_i) = 1$ then for any $B \subseteq S$, $\sigma(B) = \sum_{i=1}^{K}\sigma(B\cap E_i)$.*

Proof. First assume that σ is concave and $K = 2$. Define sets C and D by $C = (B\cap E_1)\cup E_2, D = E_1\cup(B\cap E_2)$. By concavity, $\sigma(C) \leqslant \sigma(B) + \sigma(E_2) - \sigma(B\cap E_2), \sigma(D) \leqslant \sigma(B) + \sigma(E_1) - \sigma(B\cap E_1)$ and $1 = \sigma(S) \leqslant \sigma(C) + \sigma(D) - \sigma(B)$. Substituting we obtain $1 \leqslant \sigma(B) + \sigma(E_2) - \sigma(B\cap E_2) + \sigma(B) + \sigma(E_1) - \sigma(B\cap E_1) - \sigma(B) = 1 + \sigma(B) - \sigma(B\cap E_2) - \sigma(B\cap E_1)$ or $\sigma(B\cap E_2) + \sigma(B\cap E_1) \leqslant \sigma(B)$. However the opposite inequality follows directly from concavity, which establishes the result in this case. The general result follows by repeated application of the result for $K = 2$. If σ is convex the result can be proved by reversing the inequalities in the above proof. ■

Proposition 4.5 *Let $E_1, ..., E_K$ be a non-trivial partition of S. If a decision-maker has CEU preferences, which satisfy Assumptions 4.1 and 4.2 with beliefs represented by a JP-capacity $\nu = \alpha\mu + (1-\alpha)\bar{\mu}$, where $\alpha \neq \frac{1}{2}$, and (s)he updates his/her preferences with GBU updating then the following conditions are equivalent:*

1. *(s)he is dynamically consistent,*

2. $\sum_{k=1}^{K} \mu(E_k) = 1.$

Proof of Proposition 4.5 First note that the case $\alpha = 1$ is proved by Theorem 2.1 in Eichberger, Grant, and Kelsey (2005). If $\alpha = 0$ a similar argument will establish the result. Thus we may assume $\alpha \neq 0, 1$.

2⇒1 Condition (2) implies that we may define a probability distribution over the partition $E_1, ..., E_K$ by setting $q_k = \mu(E_k)$ for $1 \leqslant k \leqslant K$. Lemma A.2 implies that for $A \subseteq S$, $\mu(A) = \sum_{k=1}^{K} \mu(A \cap E_k) = \sum_{k=1}^{K} q_k \mu_k (A \cap E_k)$, where μ_k is a capacity on E_k defined by $\mu_k(B) = \frac{\mu(B)}{q_k}$ for $B \subseteq E_k$. Thus $\nu = \sum_{k=1}^{K} q_k [\alpha\mu_k + (1-\alpha)\bar{\mu}_k]$, which implies that ν is an PAJP capacity. Hence we may apply Proposition 4.1 to deduce that the GBU update of ν conditional on E_k is $\nu_k = \alpha\mu_k(A) + (1-\alpha)\bar{\mu}_k(A)$.

Suppose that $b_k \in A(E_k)$ is preferred to a_k conditional on E_k, for $1 \leqslant k \leqslant K$. Then

$$\int u(b_k)\, d\nu_k \geqslant \int u(a_k)\, d\nu_k, \text{ for } 1 \leqslant k \leqslant K, \tag{18}$$

with at least one strict inequality. Define $b \in A(S)$, by $b(s) = b_k(s)$ if $s \in E_k$, for $1 \leqslant k \leqslant K$. We shall show that b is preferred to a in the first period, which implies dynamic consistency. Let the range of a (i.e. the set of outcomes generated by act a) be denoted by $\{x_1, ..., x_m\}$ where the outcomes have been numbered so that, $u(x_1) \geqslant u(x_2) \geqslant ... \geqslant u(x_m)$. Also define $A_i = \{s \in S : a(s) \in \{x_1, ..., x_i\}\}$. From the definition of the Choquet integral:

$\int u(a)\, d\nu = u(x_1)\nu(A_1) + \sum_{i=2}^{m} u(x_i) [\nu(A_i) - \nu(A_{i-1})]$

$= u(x_1) [\alpha\mu(A_1) + (1-\alpha)\bar{\mu}(A_1)]$

$+ \sum_{i=2}^{m} u(x_i) [\alpha\mu(A_i) + (1-\alpha)\bar{\mu}(A_i) - \alpha\mu(A_{i-1}) - (1-\alpha)\bar{\mu}(A_{i-1})]$.

By Lemma A.2 this may be rewritten as

$\sum_{k=1}^{K} u(x_1) [\alpha\mu(A_1 \cap E_k) + (1-\alpha)\bar{\mu}(A_1 \cap E_k)]$

$+ \sum_{k=1}^{K} \sum_{i=2}^{m} u(x_i) [\alpha\mu(A_i \cap E_k) + (1-\alpha)\bar{\mu}(A_i \cap E_k) - \alpha\mu(A_{i-1} \cap E_k) - (1-\alpha)\bar{\mu}(A_{i-1} \cap E_k)]$

$= \sum_{k=1}^{K} \int u(a_{E_k})\, d\nu_k$. Similarly $\int u(b)\, d\nu = \sum_{k=1}^{K} \int u(b_k)\, d\nu_k$.

Thus $\int u(b)\, d\nu \geqslant \int u(a)\, d\nu$, which implies that act a could not be chosen in the first period. It follows that the decision-maker is dynamically consistent.

1⇒2 Suppose that the decision-maker is dynamically consistent. Consider first the case $K = 2$. Since the partition is non-trivial, we may find events, A, B, C, and D such that, $E_1 = A \cup B$, $E_2 = C \cup D$, where $A \cap B = C \cap D = \varnothing$. Consider acts a, b, c, e, f and g as described in the following table:

	E_1		E_2	
	A	B	C	D
a	1	1	1	1
b	1	1	β	0
c	0	0	1	1
e	0	0	β	0
f	β	β	1	1
g	β	β	β	0

We can ensure that acts with these values exist by appropriately normalizing the utility function, (recall that X is convex). Note that $\int a_1 d\nu_1 = \int b_1 d\nu_1, \int c_1 d\nu_1 = \int e_1 d\nu_1, \int f_1 d\nu_1 = \int g_1 d\nu_1$; $\int a_2 d\nu_2 = \int c_2 d\nu_2 = \int f_2 d\nu_2$ and $\int b_2 d\nu_2 = \int e_2 d\nu_2 = \int g_2 d\nu_2$. By continuity and strong monotonicity we may choose β so that $\int a_2 d\nu_2 = \int b_2 d\nu_2$. Since $\alpha \neq 1$, $\beta > 1$. Dynamic consistency then implies that $a \sim b$, $c \sim e$ and $f \sim g$. By evaluating the Choquet integrals we find: $1 = (\beta - 1)\,\nu(C) + \nu(E_1 \cup C)$, $\nu(E_2) = \beta\nu(C)$ and $\beta\nu(E_1 \cup C) = \beta\nu(E_1) + 1 - \nu(E_1)$.
Hence $\nu(E_1 \cup C) = 1 - (\beta - 1)\,\nu(C) = 1 - \frac{\beta - 1}{\beta}\nu(E_2)$,

$\beta\nu(E_1) + 1 - \nu(E_1) = \beta - (\beta - 1)\,\nu(E_2)$,

$1 - \beta = (1 - \beta)\,\nu(E_1) + (1 - \beta)\,\nu(E_2) \Leftrightarrow \nu(E_1) + \nu(E_2) = 1$.

Thus $\alpha\mu(E_1) + (1 - \alpha)\,\bar{\mu}(E_1) + \alpha\mu(E_2) + (1 - \alpha)\,\bar{\mu}(E_2) = 1$.

Expanding $\alpha\mu(E_1) + (1 - \alpha) - (1 - \alpha)\,\mu(E_2) + \alpha\mu(E_2) + (1 - \alpha) - (1 - \alpha)\,\mu(E_1) = 1$,

or $(1 - 2\alpha) - (1 - 2\alpha)\,\mu(E_2) - (1 - 2\alpha)\,\mu(E_1) = 0$. Since $\alpha \neq \frac{1}{2}$, this implies $\mu(E_1) + \mu(E_2) = 1$.

The general case can be established as follows. If $\sum_{k=1}^{K} \mu(E_k) < 1$, then we can apply the above argument to $F_1 = E_1$ and $F_2 = \bigcup_{E \in \mathcal{E}_T, E \neq E_1}$ to deduce that dynamic consistency implies $\mu(F_1) + \mu(F_2) = 1$. By repeated application of this result we may deduce that $\sum_{k=1}^{K} \mu(E_k) = 1$.

■

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