The St. Petersburg Paradox despite risk-seeking preferences: An experimental study

James C. Cox[a], Eike B. Kroll[b,*], Vjollca Sadiraj[a] and Bodo Vogt[b]
[a] Georgia State University, Andrew Young School of Policy Studies, 14 Marietta Street, Atlanta, GA 30303, USA.
[b] Otto-von-Guericke-University Magdeburg, Empirical Economics, Universitätsplatz 2, 39106 Magdeburg, Germany

ABSTRACT
The St. Petersburg Paradox is one of the oldest violations of expected utility theory. Thus far, explanations of the paradox aim at small probabilities being perceived as zero and the boundedness of utility. This paper provides experimental results showing that neither risk attitudes nor perception of small probabilities explain the paradox. We find that even in situations where subjects are risk-seeking, and zeroing-out small probabilities supports risk taking, the St. Petersburg Paradox exists. This indicates that the paradox cannot be resolved by the arguments advanced to date.

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* Corresponding author. email: eike.kroll@ovgu.de, phone: +49 391 67 1842, fax: +49 391 67 11222.

Introduction

The St. Petersburg Paradox has attracted great interest from researchers for 300 years (Neugebauer 2010). In the original version of the St. Petersburg Game a fair coin is tossed until it comes up heads for the first time. The game pays 2^n with n indicating the number of tosses it took for the first occurrence of heads. While the St. Petersburg lottery in its original version offers an infinite expected value, people are found not to pay more than $25 for hypothetical offers to participate in the game (Hacking 1980). Various researchers have provided explanations for the paradox, but with every explanation a new version of the initial game was constructed that brought back the puzzle (Samuelson 1977). Versions of the game have been constructed that apply to all currently-popular theories of decision under risk (Cox and Sadiraj 2008).

The first explanation for the observed behavior was decreasing marginal utility of risk-averse agents (Bernoulli 1738/1954), however, the game can be constructed correcting for decreasing marginal utility and the paradox remains. Therefore, the focus shifted towards the question of infinity. Limited time was introduced as the factor putting a bound to the utility of the St. Petersburg lottery (Brito 1975; Cowen and High 1988). In contrast it was argued, that the utility of the lottery could in principle be unbounded but the offer is most probably not considered genuine (Shapley 1977) causing the decision patterns found in experimental investigations. The most straightforward solution of the paradox, however, is that utility is bounded since otherwise one can always create lotteries leading to counterintuitive solutions (Aumann 1977). But bounding utility substitutes one paradox for another: with bounded utility an agent will exhibit implausible large-stakes risk aversion (Cox and Sadiraj 2008).

To avoid infinity the St. Petersburg Game was broken down into a series of finite games, but the paradox still exists (Samuelson 1960). This fact does not indicate that infinity is the underlying cause of the paradox. Other work argues that the small probabilities cause the paradox since sufficiently small probabilities are regarded as zero (Brito 1975) or small chances for large prizes create big risks for the agents (Allais 1952; Weirich 1984). In another approach on using probabilities as an explanation for the phenomenon, more recent work introduced a new weighting function for cumulative prospect theory attempting to solve the problem of infinity (Blavatskyy 2005).

Recent experimental research has returned to the questions originally posed by Bernoulli: Is human choice behaviour in St. Petersburg lotteries (a) inconsistent with expected value theory and (b) consistent with risk aversion? Recent experiments (Cox, Sadiraj and Vogt 2009; Neugebauer 2010) have used real money payoffs and finite games in order, respectively, to provide the experiment subjects with economic motivation and the experimenters' offers of the lottery with credibility. Data from these experiments are inconsistent with risk neutrality but consistent with risk aversion, in this way appearing to provide support for Bernoulli's general conclusion about risk aversion (but not his specific conclusion about log utility).

In contrast, the present paper reports data from experiments that question whether risk aversion can explain decisions about participating in St. Petersburg lotteries. Two experiments are designed that use a modification of the St. Petersburg lotteries used in Cox, Sadiraj and Vogt (2009). In Experiment 1, the St. Petersburg lotteries are changed, so that a monetary loss is realized. That is, the number of coin tosses it takes to turn up a head determines the loss realized. In this case the participants receive a payment for participation. That means, instead of offering subjects an opportunity to pay money for participating in a lottery with positive payoffs, this experiment provides subjects with an opportunity to receive money payment for accepting an offer to participate in a lottery in which they can only lose money. Since this version translates the decisions about the St. Petersburg lotteries from gains to losses, people will make different decisions depending on whether they are risk-loving or risk averse over losses, a topic on which there is mixed evidence (Kahneman and Tversky 1979; Holt and Laury 2002).

For both, the original version of the St. Petersburg lotteries taken from Cox, Sadiraj and Vogt (2009) and the lotteries developed for this study, participants decide about tradeoffs between possible gains and losses. That means another factor besides risk attitudes could be part of the explanation, which is loss aversion. In order to control for loss aversion as a possible source for explaining the behavioral pattern in decisions about St. Petersburg lotteries, a second experiment is designed. In Experiment 2, the same lotteries as in Experiment 1 are used, but with the difference that the payoffs are substituted by waiting times. In a first step the experiment elicits subjects' preferences for waiting times, which turn out to be risk-seeking. In a second step, the St. Petersburg lotteries are designed using waiting time in a way that places possible waiting-time "payoffs" within the part of the utility function over time that subjects have revealed to be convex.

Data from experiments reported in this paper are inconsistent with the conclusion that risk aversion explains patterns observed for decisions about St. Petersburg lotteries in experiments. Furthermore, it is argued that following the experiment results of this study, none of the proposed explanations for the St. Petersburg Paradox can explain the behavior in these experimental settings.

Experiment 1

The experiment was conducted with 15 students from the Otto-von-Guericke-University Magdeburg from different fields of study. With the electronic invitation the participants were informed that the experiment would be performed on two different days. Furthermore, they were informed that real losses could result from the experiment, while these losses could be avoided by individuals' decisions. Subjects were also informed that, in the event that a participant realized a loss, she would have to pay the lost amount from her own pocket money.

During the first meeting, the participants received a show-up fee of 10 euros. Furthermore, the information from the invitation was read aloud to the subjects. After receiving the show-up fee, the participants filled out a form stating they were fully aware of the information provided by the experimenter and they agreed to pay any losses that might occur from participation in the experiment. It was explained that if it turned out that the experimenters realized a surplus from this experiment they would use the money for costs of other experiments.

The actual experiment on decisions about St. Petersburg lotteries was performed during the second meeting. The experiment consisted of the same lotteries used in Cox, Sadiraj and Vogt (2009) with the only difference being that all payoffs were multiplied by the factor -1. For a total of 9 lotteries, the participants could choose between participating in the lottery or not. The lotteries differed in the maximum number of coin tosses n (with n = 1,2,…,9). If a subject chooses to participate in a lottery, she received n euros. Then a coin was tossed until either a head occurred or the maximum number of tosses for that lottery was reached. If the coin turned up heads on the i-th toss, the participant was required to pay 2^i euros. If the coin did not turn up heads on any of the n tosses, the participant was not required to pay anything and her payoff remained at n euros. In case the subject chose not to participate in the lottery, she received no payment. The lotteries with possible outcomes are presented in Table 1. After the participants made their decisions for all of the 9 lotteries, for each participant one lottery was randomly chosen for realization.

Head occurs the first time at toss no.:	Probability	Payoff [in euros]
1	0,5	-2
2	0,25	-4
3	0,125	-8
4	0,0625	-16
5	0,03125	-32
6	0,015625	-64
7	0,0078125	-128
8	0,0039062	-256
9	0,0019531	-512
Not at all		+/-0

Table 1: St. Petersburg Lotteries for monetary losses

Result 1

The analysis of the data from our version of the St. Petersburg lotteries is adapted from the procedures used in Cox, Sadiraj and Vogt (2009) in order to make comparisons. The analysis focuses on the proportion of subjects rejecting the St. Petersburg lottery for the maximum number of coin tosses n. For the original form of the St. Petersburg lotteries it was found that the proportion of subjects rejecting a lottery increased with increasing numbers of maximum tosses of the coin. Since the version of the lottery used in this experiment involves losses, subjects will not reject any of the lotteries if they have risk-loving preferences over monetary losses.

The finding of the experiment, however, shows that the pattern of rejecting St. Petersburg lotteries with monetary gains and losses is essentially the same. While the proportion of subjects rejecting the lotteries is higher on average when losses are involved, the pattern of low levels of rejecting lotteries with small n and high levels of rejection for lotteries with high n remains the same. Therefore, there is a pattern to be found when subjects make decisions about St. Petersburg lotteries that is robust to whether monetary returns from the lotteries are gains or losses.

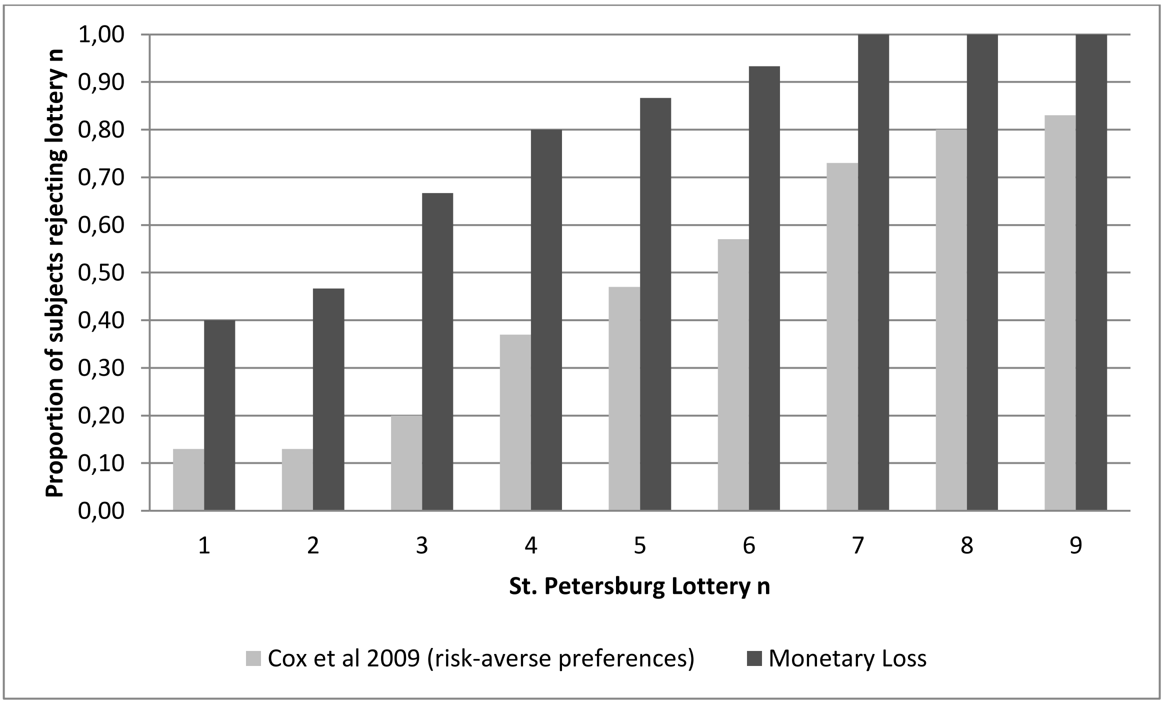

Figure 1: Comparison of classic St. Petersburg Lotteries with monetary losses

Experiment 2

The experiment was conducted with 74 students from the Otto-von-Guericke-University Magdeburg from different fields of study. The experiment was conducted in a laboratory environment. The participants were divided into three groups, with one eliciting risk-preferences for waiting time (Treatment 1, 36 participants), and two groups playing St. Petersburg Lotteries with different base waiting times (Treatment 2, 25 participants and Treatment 3, 24 participants). All participants received a show-up fee of 8 euros before experiment instructions were handed out. Furthermore, it was made clear that there would be no other monetary rewards from participation in the experiment other than the consequences described in the experiment instructions.

Risk Preference for Waiting Time

To elicit preferences over waiting time, participants were asked to choose between two lotteries in each of 10 pairs of lotteries where "payoffs" were determined as waiting time. After choices were made, one choice pair was randomly selected for realization. In our experiment participants were paid a show-up fee of 6 euros at the beginning of the experiment and told that their decisions would determine a waiting time in the laboratory. This waiting time started after all decisions were made and the chosen lotteries were played out. The participants spent this waiting time in a laboratory cabin without any communication devices or books.

The options were presented in a format similar to the one used in Holt and Laury; 2002), where option A offered less risk but a higher sure waiting time (with a waiting time of either 30 or 40 minutes) and option B offered a higher risk but the chance of a much smaller waiting time (with a waiting time of either 5 or 60 minutes). The probabilities of the favorable outcome stayed the same for both options but varied between .1 and 1.0 as shown in 0 Therefore, risk preferences for waiting time could be elicited for each participant by the row in which option B was chosen for the first time. If that point was in row 4 or earlier the choice pattern would indicate risk-seeking behavior, if it was in row 6 or later the choice pattern would risk-averse behavior. The risk attitude for subjects switching to option B in row 5 cannot be identified since they could be either slightly risk-averse, risk-neutral or slightly risk-seeking.

No.	Option A	Option B	Expected Value difference
1	{.1, 30, .9, 40}	{.1, 5, .9, 60}	-15.5
2	{.2, 30, .8, 40}	{.2, 5, .8, 60}	-11
3	{.3, 30, .7, 40}	{.3, 5, .7, 60}	-6.5
4	{.4, 30, .6, 40}	{.4, 5, .6, 60}	-2
5	{.5, 30, .5, 40}	{.5, 5, .5, 60}	2.5
6	{.6, 30, .4, 40}	{.6, 5, .4, 60}	7
7	{.7, 30, .3, 40}	{.7, 5, .3, 60}	11.5
8	{.8, 30, .2, 40}	{.8, 5, .2, 60}	16
9	{.9, 30, .1, 40}	{.9, 5, .1, 60}	20.5
10	{1.0, 30, 0.0, 40}	{1.0, 5, 0.0, 60}	25

Table 2: Lottery choices determining waiting time

After the choices were made, the experimenter drew a ball from a bingo cage with balls labeled from 1 to 10, which determined which decision was selected. Then, the lottery the participant chose for the decision in that row was realized and the waiting time started.

St. Petersburg Game

In a second step participants were offered a series of St. Petersburg lotteries. All subjects had a base waiting time (Treatment 2, 10 minutes; Treatment 3, 45 minutes) and were offered an opportunity to participate in a lottery where this waiting time could be reduced or increased depending on their decision and the outcome of the lottery. This lottery was designed analogous to the St. Petersburg lottery used in Cox, Sadiraj and Vogt (2009). For participation in the lottery the waiting time was reduced by n minutes and a coin was tossed until a head occurred, with a maximum of n tosses. If head occurred on the i-th toss, the waiting time was increased by 2^i minutes. Each participant was offered 9 lotteries with only one choice being randomly selected for payoff, with the lotteries differing by the maximum number of tosses n (see 0). For example, suppose decision 3 was randomly picked to be realized for a subject. If the subject chose not to play this game, the resulting waiting time was at the base waiting time of 10 (or 45) minutes. If the subject chose to play the game, the base waiting time was reduced by 3 minutes to 7 (or 42) minutes. Then, a coin was tossed. If it came up heads at the first toss the waiting time was increased by 2 minutes to 9 (or 44) minutes. If it came up heads at the second toss, the waiting time was increased by 4 minutes to 11 (or 46) minutes. If it came up heads at the third toss, the waiting time was increased by 8 minutes to 15 (or 50) minutes. If the coin did not come up heads at any of the three tosses, the waiting time remained at 7 (or 42) minutes.

Heads occurs the first time at toss no.:	Probability	Additional waiting time [in minutes]
1	0,5	2
2	0,25	4
3	0,125	8
4	0,0625	16
5	0,03125	32
6	0,015625	64
7	0,0078125	128
8	0,0039062	256
9	0,0019531	512
Not at all		+/-0

Table 3: St. Petersburg Lotteries for waiting time.

After the participants made their choices, the experimenter drew a ball from a bingo cage numbered from 1 through 9 to select which choice would be realized. If the participant chose not to play the game offered, the base waiting time was realized and started immediately. If the participant chose to play the game, the experimenter tossed the coin as described above and determined the actual waiting time. All participants spent their waiting time in a laboratory cabin without communication devices or other kinds of entertainment possibility. To control for reference-dependence of preferences (Köszegi and Rabin 2007; Farber 2008), we ran two treatments with different base waiting times of 10 and 45 minutes.

Results 2

Risk Preferences for Waiting Time

As described above, subjects can be sorted as risk-seeking or risk-averse for choices on waiting time by looking at the first row in which option B is chosen. In 0 it can be seen from the differences in expected values that risk-seeking individuals would choose option B for the first time in row 4 or earlier, while the switching point from option A to option B would be in row 5 or later for risk-averse subjects. The frequencies for rows in which subjects switched to option B are reported in 0. That means, a subject who chooses option A in rows 1 through 3 and chooses option B in rows 4 through 10 is noted in column 4, while a subject who chooses option A in rows 1 through 4 and then switches to option B is noted in column 5. From the expected value differences in 0 it can be seen that subjects listed in columns 1 through 4 are risk-seeking and subjects listed in columns 6 through 10 are risk-averse. Subjects who are listed in column 5 cannot be clearly identified, as noted above. One subject was excluded from the analysis, because of switching from option A to B and back to A several times.

Risk Preference	risk-seeking					risk-averse			
Row of first choosing option B	1	2	3	4	5	6	7	8-10	∑
Frequency	1	1	6	19	4	2	2	1	36

Table 4: Risk preference for waiting time

The data set shows 27 subjects with risk-seeking behavior while 9 subjects made choices showing risk-averse behavior. Therefore we conclude that people mainly show risk-seeking behavior when making decisions about waiting time, where the outcome is subject to risk (1%-level, Binomial-Test).

St. Petersburg Game

Knowing the results from elicitation of risk preferences over waiting time, one would predict that a high majority of subjects would play all of the offered St. Petersburg lotteries for waiting time. The expected value of the offered gambles on waiting times is equal to the base waiting time. Therefore, a risk-seeking individual would choose to participate in all offered gambles. The results of the St. Petersburg lotteries show, that while individuals do participate in the gambles for small reductions of the base waiting time, they do not for higher possible reductions of the base waiting time. Therefore, decision patterns are similar to the ones found in the real-payoff experiment with St. Petersburg lotteries with positive monetary payoffs (Cox, Sadiraj and Vogt 2009).

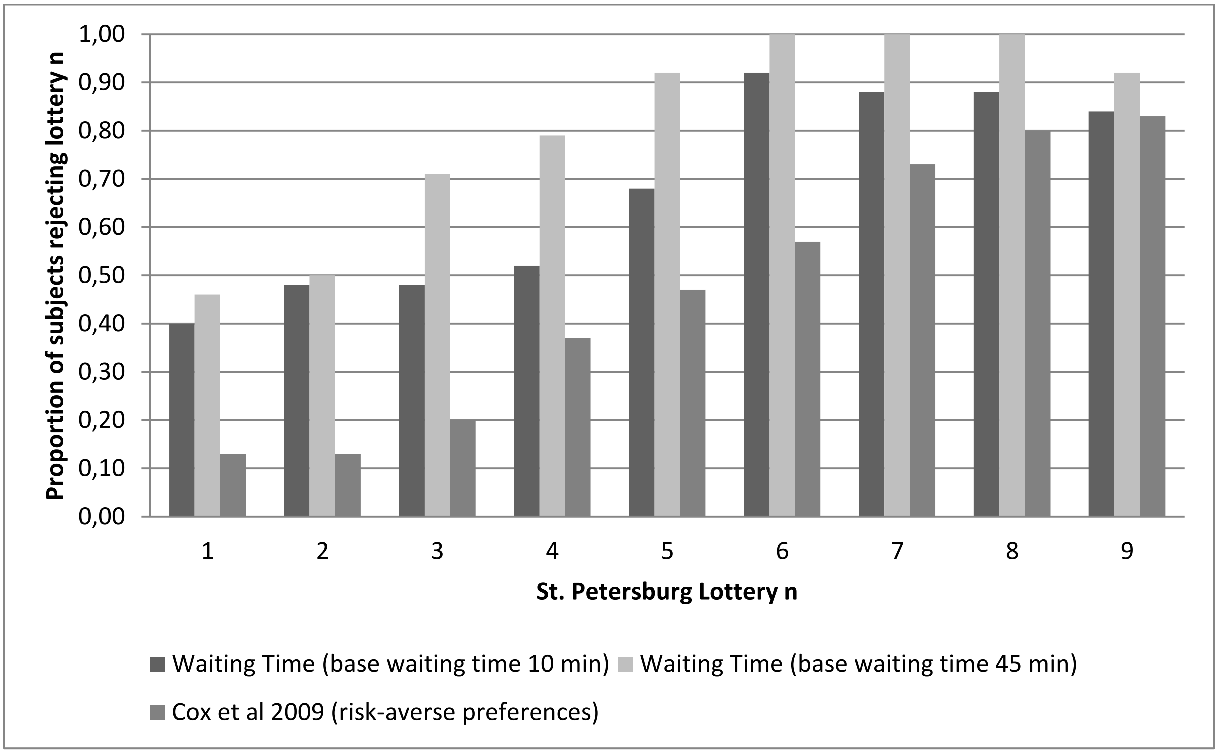

Figure 2: Comparison of rejecting St. Petersburg Lotteries of risk-averse and risk-seeking preferences

In the treatment with a base waiting time of 10 minutes, 2 of the 25 participants choose never to play the game, while the rest mostly starts playing the first game, but switches to answering ‘no’ along the line. None of the participants chooses to play all offered games.

The data from the treatment with a base waiting time of 45 minutes yield similar results. There are a lower number of subjects who play lotteries with high reductions of the base waiting time than observed in the first treatment. However, the difference is not significant on a statistical level.

Conclusion

Initially the St. Petersburg Paradox was designed to point out a weakness of expected value theory showing that decision makers are not risk-neutral. As a result the idea of using utilities rather than monetary payoffs and introducing decreasing marginal utility for money was developed. From that point on, economists have focused on developing theories of decision under risk that account for risk-averse behavior as initially reported for hypothetical experiments with the infinite horizon St. Petersburg lottery. Various possibilities of modeling risk-averse behavior in decisions over risky prospects have been proposed in the literature. Risk-averse behavior can be incorporated by nonlinear transformation of payoffs (Bernoulli 1738/1954; Pratt 1964; Arrow 1971), nonlinear transformation of probabilities (Yaari 1987), or nonlinear transformation of both payoffs and probabilities (Quiggin 1993; Kahneman and Tversky 1992). But there are versions of St. Petersburg lotteries that produce paradoxes for all of these theories if they are defined on unbounded domains (Cox and Sadiraj 2008). If, instead, the domain is bounded then the paradox does not exist; for example, if the largest credible prize offered in a St. Petersburg lottery is 35 billion euros then the expected value of the lottery is about 35 euros and it would not be paradoxical if people were unwilling to pay large amounts to play.

Although the original infinite horizon St. Petersburg lottery cannot credibly be offered to anyone in a real-payoff experiment, finite versions are still of interest for elicitation of risk preferences. Recent real-money experiments with positive-payoff St. Petersburg lotteries produced data that are inconsistent with risk neutrality but consistent with risk aversion (Cox, Sadiraj and Vogt 2009; Neugebauer 2010). One experiment in this paper shows that St. Petersburg lotteries elicit similar behavior on both monetary loss and gain domains. Even more strikingly, the experiment on the loss domain of waiting times produces similar St. Petersburg lottery behavior even though most subjects reveal risk-loving preferences in this domain. Therefore, risk preferences are not a conclusive explanation for the behavioral pattern found for St. Petersburg lotteries.

Risk aversion is, however, not the only solution to the paradox as proposed by the literature. There is also the problem of infinity (Brito 1975; Cowen and High 1988) which is associated with the original form of the game. Therefore, when playing St. Petersburg lotteries with real consequences a series of finite St. Petersburg lotteries is used. Furthermore, the question was raised whether participants would regard the offer of the original game as genuine (Shapley 1977). Both types of explanation do not hold for the findings in this paper. While infinity is not the problem in this type of game, it is not possible to control whether participants regarded the offer as genuine or not. However, both explanations would create a behavioral pattern in the experiment of this paper that is the opposite of what was observed. For the point of infinity, if subjects would not believe the experimenter to toss a coin as often as proposed, it would be a safe bet to play the lotteries with a higher number of maximum tosses. Furthermore, if subjects would regard an offer as not genuine it is reasonable to assume that high monetary losses or the longer waiting times would not be realized and, therefore, a subject playing the lotteries with relatively low maximum number of coin tosses n would definitely play the lotteries with relatively high n. That is not what is observed in the experiment.

Another argument explaining the behavioral patterns in St. Petersburg lotteries is that utility is bounded (Aumann 1977). For the version of the game proposed in this paper, an upper bound on utility is the waiting time of zero which is not realized for any of the lotteries proposed. Since the waiting time in this experiment can be interpreted as a loss, the bound of utility would have to occur at the maximum loss that can be perceived. If such a bound exists it would induce the same behavioral pattern as described for the problem of infinity. Therefore, bounded utility cannot explain the same behavioral pattern for the original version of the lottery with monetary gains and the lotteries used in this paper with monetary losses and waiting times as possible consequences of the St. Petersburg lotteries.

Other papers argued that very small probabilities are regarded as zero (Brito 1975) or that small probabilities for high wins result in a high risk for the decision maker (Allais 1952; Weirich 1984). These explanations also do not help explaining our results.

In conclusion it can be noted that various conjectures have been advanced to explain behavior with St. Petersburg lotteries. However, none of the conjectures can explain what is observed in our experiments with decisions for real monetary and waiting time outcomes. Whether weighting functions are used for payoffs, probabilities or both, the existing theories provide models of risk preferences for subjects. An implication of these models is that behavior in St. Petersburg lotteries depends on risk preferences. Therefore, none of the theories can explain the results of this paper.

References

Allais, M. (1952). Le comportement de l'homme rationnel devant le risqué: Critique des postulats et axioms de l'ecole americaine. Econometrica, 21(4), 503-546.

Arrow, K. J. (1971). Essays in the theory of risk-bearing. Chicago: Markham Pub. Co.

Aumann, R. J. (1977). The St. Petersburg Paradox: A discussion of some recent comments. Journal of Economic Theory, 14(2), 443- 445.

Bernoulli, D. (1738/1954). Exposition of a New Theory on the Measurement of Risk. Econometrica, 22(1), 23-36.

Blavatskyy, P. R. (2005). Back to the St. Petersburg Paradox? Management Science, 51(4), 677-678.

Brito, D. L. (1975). Becker's Theory of the allocation of time and the St. Petersburg Paradox. Journal of Economic Theory, 10(1), 123-126.

Cowen, T., & High, J. (1988). Time, Bounded Utility, and the St. Petersburg Paradox. Theory and Decision, 25(3), 219–233.

Cox, J. C., & Sadiraj, V. (2008). Risky Decisions in the Large and in the Small: Theory and Experiment. In J. C. Cox & G. W. Harrison (Eds.), Risk aversion in experiments (Vol. 12, pp. 9–40). Bingley: JAI Press.

Cox, J. C., Sadiraj, V., & Vogt, B. (2009). On the empirical relevance of St. Petersburg lotteries. Economics Bulletin, 29(1), 221-227.

Farber, H. S. (2008). Reference-Dependent Preferences and Labor Supply: The Case of New York City Taxi Drivers. American Economic Review, 98(3), 1069–1082.

Hacking, I. (1980). Strange Expectations. Philosophy of Science, 47(4), 562-567.

Holt, C. A., & Laury, S. K. (2002). Risk Aversion and Incentive Effects. American Economic Review, 92(5), 1644-1655.

Kahneman, D., & Tversky, A. (1979). Prospect Theory: An analysis of decision under risk. Econometrica, 47(2), 263-292.

Kahneman, D., & Tversky, A. (1992). Advances in prospect theory: Cumulative representation of uncertainty. Journal of Risk and Uncertainty, 5(1), 297-323.

Köszegi, B., & Rabin, M. (2007). Reference-Dependent risk attitudes. American Economic Review, 97(4), 1047-1073.

Neugebauer, T. (2010). Moral Impossibilitiy in the St. Petersburg Paradox: A Literature Survey and Experimental Evidence. LSF Research Working Paper Series, 10(174), 1–43.

Pratt, J. W. (1964). Risk Aversion in the Small and in the Large. Econometrica, 32(1-2), 122–136.

Quiggin, J. (1993). Generalized Expected Utility Theory. The Rank-Dependent Model. Boston: Kluwer Academic Publishers.

Samuelson, P. A. (1960). The St. Petersburg Paradox as a Divergent Double Limit. International Economic Review, 1(1), 31-37.

Samuelson, P. A. (1977). St. Petersburg Paradoxes: Defanged, Dissected, and Historically Described. Journal of Economic Literature, 15(1), 24-55.

Shapley, L. S. (1977). The St. Petersburg Paradox: A Con Game? Journal of Economic Theory, 14(2), 439-442.

Weirich, P. (1984). The St. Petersburg Gamble and Risk. Theory and Decision, 17(2), 193-202.

Yaari, M. E. (1987). The Dual Theory of choice under risk. Econometrica, 55(1), 95-115.