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1 Introduction

This paper introduces a kernel operator that (1) transforms probability domains, and (2) generates sample paths for confidence representation initiated by convex random source sets of priors induced by ambiguity. It also provides several applications of the results. The operator is motivated by compensating probability factors generated by deviations of a subjective probability measure from an equivalent martingale measure. See Harrison and Kreps (1979) and Sundaram (1997) for equivalent martingale measures, and (Fellner, 1961, pg. 672) for compensating probabilities. The magnitude of deviations are controlled by the curvature of probability weighting functions. See Wu and Gonzalez (1996); Gonzalez and Wu (1999). Our behavioural theory extends Tversky and Wakker (1995) who characterized the shape of probability weighting functions as being dispositive of the impact of an event. For example, they considered the steepness of a classic inverse S-shaped probability weighting function (PWF) near its endpoints. And introduced the concept of *bounded subadditivity* to explain the phenomenon of an impact event in which a subject transforms impossibility into possibility, and possibility into certainty, in regions near the extremes of the PWF. That event makes a possibility more or less likely in the ”middle” portion of the PWF. This paper introduces an operator or kernel function (that depends on a PWF) that shows how Tversky-Wakker subjects transforms probability domains. For instance, it shows how a subject transforms loss probability domain into hope of gain to explain risk seeking behaviour. And how gain probability domain is transformed into fear of loss to explain risk aversion. It extends the literature by showing how the operator also generates sample paths for confidence. In particular, in Lemma 2.3 below, we show how loss aversion is akin to a Langevin type frictional force that induce mean reversion in behaviour typically modeled by Ornstein-Uhlenbeck processes.

More recent, Abdellaoui et al. (2011) introduced a model in which they treated sources of uncertainty as an algebra of events. In particular, they posited: "The function w_S , carrying subjective probabilities to decision weights, is called the source function". And they state unequivocally that source functions "represent deviations from rational behavior". They also report that a rich variety of ambiguous attitudes were found between and within person. Almost all of those results, or a reasonable facsimile of them, are predicted by our model. Here, the source of uncertainty is reflected by a convex random set of prior probabilities for unknown states. Theoretically, these random source sets are comprised of elementary events. So they are consistent with "sources of uncertainty as algebra of events". We address that issue in Lemma 2.1 which establishes a nexus between algebra of events, ambiguity, and our convex random source set of priors. In subsection 3.2 in this paper, we introduce Lemma 3.2 which shows how our confidence kernel, induced by ambiguous random set or priors, extends "source functions" to decision weights. Specifically, our confidence kernel is based on the area under the "source function" or probability weighting function/curve adjusted for loss gain probability spread relative to an equivalent martingale measure. So it naturally extends the source function approach to ambiguity. By contrast, extant confidence indexes are survey driven, ie, Shiller (2000) and Manski (2004); or derived from comparatively ad hoc computer driven principal components analysis, ie, Baker and Wurgler (2007). To the best of our knowledge the confidence operator, and sample path representation for confidence introduced in this paper are new[1]. In Corollary 2.6 we also make the case for the use of conjugate priors as a mechanism for reducing discrepancy in the Gilboa and Schmeidler (1989) set of priors. In the sequel we provide several applications for our theory, and conduct a simple weak hypothesis test which upheld the source set hypothesis.

The rest of the paper proceeds as follows. In section 2 we introduce our model. In

[1]We also note that the approach taken in this paper is distinguished from extant models of ambiguity introduced by the Italian school or otherwise. See eg, Klibanoff et al. (2005) (smooth ambiguity); Maccheroni et al. (2006) (variational model of that captures ambiguity); Cerreia-Vioglio et al. (2011) (uncertainty averse preferences); Cerreia-Vioglio et al. (2011) (rational model of ambiguity without certainty independence and uncertainty aversion)

section 3 we use a simple example to explain our theory, and provide several applications of our theory ranging from constructing confidence preferences, simulation, programmed trading, the role of confidence in bubbles, crashes and volatility in financial markets. We conclude with perspectives for further research in section 4.

2 The Model

Specifically, let p^* be a fixed point probability that separates loss and gain domains; and let $\mathcal{P}_\ell \triangleq [0, p^*]$ and $\mathcal{P}_g \triangleq (p^*, 1]$ be loss and gain probability domains as indicated. So that the entire domain is $\mathcal{P} = \mathcal{P}_\ell \cup \mathcal{P}_g$. Let $w(p)$ be a probability weighting function (PWF), and p be an equivalent martingale measure. The confidence index from loss to gain domain is a real valued mapping defined by

$$K : \mathcal{P}_\ell \times \mathcal{P}_g \to [-1, 1] \tag{1}$$

$$K(p_\ell, p_g) = \int_{p_\ell}^{p_g} [w(p) - p]dp = \int_{p_\ell}^{p_g} w(p)dp - \frac{1}{2}(p_g^2 - p_\ell^2), \quad (p_\ell, p_g) \in \mathcal{P}_\ell \times \mathcal{P}_g \tag{2}$$

We note that that kernel can be transformed even further so that it is singular at the fixed point p^* as follows:

$$\hat{K}(p_\ell, p_g) = \frac{K(p_\ell, p_g)}{p_g - p_\ell} = \frac{1}{p_g - p_\ell} \int_{p_\ell}^{p_g} w(p)dp - \frac{1}{2}(p_g + p_\ell) \tag{3}$$

The kernel accommodates any Lebesgue integrable PWF compared to any linear probability scheme. See e.g., Prelec (1998) and Luce (2001) for axioms on PWF, and Machina (1982) for linear probability schemes. Evidently, $\hat{K}$ is an averaging operator induced by K. The estimation characteristics of these kernels are outside the scope of this paper. The interested reader is referred to the exposition in Stein (2010). Let $\mathfrak{T}$ be a partially ordered index set on probability domains, and $\mathfrak{T}_\ell$ and $\mathfrak{T}_g$ be subsets of $\mathfrak{T}$ for indexed loss and indexed gain probabilities, respectively. So that

$$\mathfrak{T} = \mathfrak{T}_\ell \cup \mathfrak{T}_g \tag{4}$$

For example, for $\ell \in \mathfrak{T}_\ell$ and $g \in \mathfrak{T}_g$ if $\ell = 1, \ldots, m; \quad g = 1, \ldots, r$ the index $\mathfrak{T}$ gives rise to a $m \times r$ matrix operator $K = [K(p_\ell, p_g)]$. The "adjoint matrix" $K^* = [K^*(p_g, p_\ell)] = -[K(p_\ell, p_g)]^T$. So K transforms gain domain into loss domain–implying fear of loss, or risk aversion, for prior probability p_ℓ. While K^* is an *Euclidean motion* that transforms loss domain into hope of gain from risk seeking for prior gain probability p_g. Thus, K^* captures Yaari (1987) "reversal of the roles of probabilities and payments", ie, the preference reversal phenomenon in gambles first reported by Lichtenstein and Slovic (1973). Moreover, K and K^* are generated (in part) by prior probability beliefs consistent with Gilboa and Schmeidler (1989). If V_g and V_ℓ are gain loss domains, respectively, then: $K : V_g \to V_\ell$ and $K^* : V_\ell \to V_g$. Let $f = \{(x_1, p_1), \ldots, (\mathrm{x}_n, p_n)\}$ be a lottery in which outcome x has associated probability p of occurrence and $n = m + r$. By rank ordering outcomes relative to a reference point, the probability distribution $(p_1, \ldots, p_n)$ is ineluctably separated by p^*. Thus, our index allows us to produce a numerical score for a subject's confidence transformation of loss and or gain domains accordingly.

2.1 Stochastic confidence kernel induced by random domains

We now introduce the following:

Definition 2.1 (Stochastic kernel)**.** Let $M = M_\ell \cup M_g$ be a space of loss gain probability measures; $\mathfrak{S}$ be the σ-field of Borel subsets of M, and μ be a measure on M. The sub-σ fields $\mathfrak{S}_{|M_\ell}$, $\mathfrak{S}_{|M_g}$ are the restrictions to loss and gain domains. A stochastic kernel K is a real [complex] valued mapping $K : M \times M \to Y$ such that for point $p \in M$ and a set $B \in \mathfrak{S}$ it has the properties: (i) for p fixed it is a distribution in B; and (ii) for B fixed it is a Baire function in x.

Remark 2.1. This definition is adapted from (Feller, 1970, pg. 221).

□

2.1.1 Random set topology for ambiguity

Like (Tversky and Wakker, 1995, pg. 1258), we assume that a state ω occurs but a subject is uncertain about which one. For example, given a sample space or set of states of nature Ω, if $B_g \in \mathfrak{S}_{|M_g}$, and $\tilde{B}_\ell = \{p_\ell |\ p_\ell : \Omega \to M_\ell\}$, then $K(p_\ell(\omega), B_g)$ is a stochastic kernel controlled by the random set of priors $\tilde{B}_\ell$. This is functionally equivalent to Chateaunerf and Faro (2012) fuzzy set of priors. See Zadeh (1968). Given a function f (not the lottery above) in the domain $\mathcal{D}(K)$ of K, we have the random integral equation

$$C^{p_\ell}(p_g, \omega) = C^{p_\ell(\omega)}(p_g) = (Kf)(p_\ell, \omega) = \int_{B_g} K(p_\ell, \omega, y) f(y) \mu(dy), \ \ p_g \in B_g \tag{5}$$

Thus $C^{p_\ell}(p_g, \omega)$ is the transformation of $f \in \mathcal{D}(K)$ into a distribution or trajectory over B_g anchored at $p_\ell(\omega)$. In keeping with Abdellaoui et al. (2011) source function theory[2], we state the following

Lemma 2.1 (Algebra of convex random set of priors).
M_ℓ is an algebra of the convex random set of priors $\tilde{B}_\ell$ such that if $\tilde{B}_{\ell,k}$, $k = 1, \ldots, m$ is a finite cover for M_ℓ, then

$$M_\ell = \Big(\bigcup_{k=1}^{m} \tilde{B}_{\ell,k}\Big) \bigcap \tilde{B}_\ell \tag{6}$$

Proof. See (Gikhman and Skorokhod, 1969, pp. 41-42). □

The intuition of the lemma, in which $p_\ell \in \tilde{B}_\ell \Rightarrow p_\ell \in M_\ell$, is as follows. Each [unobserved] covering set $\tilde{B}_{\ell,k}$ contains an elementary event of interest to our subject. In effect, the

[2] Fedel et al. (2011) also introduced algebras of events to characterize probabilistic reasoning. But their context is different from Abdellaoui et al. (2011).

covering sets are like the balls that cover Ellsberg (1961) urn. In the simplest case, if the covering sets $\tilde{B}_{\ell,k}$ were disjoint, then at most [s]he could surmise the existence of an index k_0, say, such that $p_\ell \in \tilde{B}_\ell \cap \tilde{B}_{\ell,k_0}$. In which case, we have a [random] prior probability $p_{\ell_0} = P\{\tilde{B}_\ell \cap \tilde{B}_{\ell,k_0}\}$. However, [s]he does not know k_0 so [s]he is faced with ambiguity. If the covering sets are not disjoint, then p_ℓ lies in several covering sets. So we have a subset of unknown indexes $k_0, k_1, \ldots, k_m$ for the possible covering sets in which p_ℓ lies. A subject endowed with Machina and Schmeidler (1992) probabilistic sophistication may opt to use entropy methods to discern a prior distribution.

2.2 Sample function from field of confidence

With the foregoing definition of ambiguity in mind, we proceed as follows.

Definition 2.2 (**Random field of confidence and term structure**).
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $M = M_\ell \cup M_g$ be a space of loss gain probability measures, and $K : M \times M \to Y$ be a kernel function with range in Y. Thus we write $(P, \omega) \equiv P(\omega) \in M$. Let $\mathfrak{Y}$ be the σ field of Borel subsets of Y. Let K be $\mathcal{F}$ measurable for every point $(p_\ell(\omega), p_g(\omega))$. For $B_g \subseteq M_g$, and a convex set of prior loss probabilities $\tilde{B}_\ell = \{p_\ell |\ p_\ell : \Omega \to M_\ell\}$, define the confidence function, with respect to a measure π on M

$$C_\ell^{p_\ell}(p_g, \omega) = (Kf)(p_\ell, \omega) = \int_{p_g \in B_g} K((p_\ell, \omega),\ p_g) f(p_g) \pi(dp_g), \quad g \in \{1, \ldots, r\} \tag{7}$$

For some set $E \in \mathfrak{Y}$ we have $C_\ell^{p_\ell}(p_g, \omega) \in E$. For $g = 1, \ldots, r$ let $\mu_{1,\ldots,r}$ be a measure on Y^r such that the joint distribution on the probability measure space $(Y^r, \mathfrak{Y}^r, \mu_{1,\ldots,r})$ is given by

$$P\{\omega; C_\ell^{p_\ell}(p_1, \omega) \in E, \ldots, C_\ell^{p_\ell}(p_r, \omega) \in E\} = \mu_{1,\ldots,r}(E^r) \tag{8}$$

Then $(\Omega, \mathcal{F}, P)$, $\{C_\ell^{p_\ell}(p_g, \omega)\}$ is a random field representation of the measure $\mu_{1,\ldots,r}$. Moreover,

$C_\ell^{p_\ell}(p_g,\omega),\quad \ell=1,\ldots,m$ is a "term structure" field relative to the "term" p_g. The same definitions hold for the kernel function $K^*=-K^T$, $B_\ell\subseteq M_\ell$, the convex set of prior gain probabilities $\tilde{B}_g=\{p_g|\ p_g:\Omega\to M_g\}$, and $h\in\mathcal{D}(K^*)$ for

$$C_g^{*p_g}(p_\ell,\omega)=(K^*h)(p_g,\omega)=\int_{p_\ell\in B_\ell}K^*(p_\ell,\ (p_g,\omega))h(p_\ell)\pi(dp_\ell),\quad \ell\in\{1,\ldots,m\} \tag{9}$$

Whereupon $C_g^{*p_g}(p_\ell,\omega),\quad g=1,\ldots,r$ is a "term structure" field relative to the "term" p_ℓ.

$\square$

Remark 2.2. This definition is motivated by (Gikhman and Skorokhod, 1969, pp. 107-108). Equations (7) and (9) are random integral equations of Volterra type of the first kind with random initial value. See (Bharucha-Reid, 1972, pp. 135, 140, 148).

$\square$

In Definition 2.2 above, $C_\ell^{p_\ell}(p_g,\omega)$ is a sample function[3] from the space of confidence trajectories over gain probability domains–for given *prior loss probability* p_ℓ in the random source set $\tilde{B}_\ell(\omega)$. We state the following

Theorem 2.2 (Confidence Representation). *Let M be the space of probability measures; $\mathfrak{S}$ be the σ-field of Borel subsets of M ; π be a measure on M; and $I(\omega)=[\hat{p}(\omega),p]\subseteq M$ be a random interval domain induced by a random prior probability $\hat{p}(\omega)$ attributable to ambiguity aversion. Let $(\Omega,\mathcal{F},P)$ be a probability space, and*

$$C(p,\omega)=(Kf)(p,\omega)=\int_{I(\omega)}K(p,y)f(y)\mu(dy)$$

be a mean-square continuous (in p) sample function from a random field of confidence in

[3] Chateauneuf and Faro (2009) introduced a "confidence function" that is different from ours.

Hilbert space $L^2(M, \mathfrak{S}, \pi)$. The confidence kernel $K(p, y)$ is defined on $I(\omega) \times I(\omega)$. Let

$$\Sigma(p_1, p_2) = E[C(p_1, \omega)C(p_2, \omega)|\ \mathcal{F}] = \int_{I(\omega)} C(p_1, \omega)\bar{C}(p_2, \omega)dP(\omega)\ a.s.\ P$$

be the covariance function of C, where $\bar{C}$ is the complex conjugate in the event C is a complex function for inner products on L^2. Let $\{\xi_n(\omega)\}_{n=1}^{\infty}$ be an orthogonal sequence of $\mathcal{F}$-measureable random variables such that $E[|\xi_n(\omega)|^2|\ \mathcal{F}] = \lambda_n$ where λ_n is an eigenvalue of $\Sigma(p_1, p_2)$, with corresponding eigenfunction $\phi(p)$. Then we have

$$C(p, \omega) = \sum_{n=1}^{\infty} \xi_n(\omega)\phi(p, \omega)\ \ a.s\ P \tag{10}$$

Proof. See Appendix A. □

Remark 2.3. A similar representation is given in (Bharucha-Reid, 1972, pp. 145-146). Except there, the initial value for the integral domain is not random, and the covariance function and eigenvalues λ'_n pertain to f in the domain $\mathcal{D}(K)$, in the representation $C(p, \omega) = (Kf)(p, \omega)$ in which case

$$C(p, \omega) = \sum_{n=1}^{\infty} \sqrt{\lambda'_n}\psi_n(\omega)\phi_n(p) \tag{11}$$

and f has a similar representation for some function $\psi_n(p) \in \mathcal{D}(K)$.

□

Figure 1 on page 11 depicts a sample function from the random field of confidence over the random interval $I(\omega) = [p_\ell(\omega), p_g]$. The initial confidence level $C_\ell^{p_\ell}(p_g^L)$ over gain domains starts at the lowest level for gain probability p_g^L moving from left to right. By contrast, Figure 2 depicts a sample function over the random interval $I(\omega) = [p_\ell, p_g(\omega]$. It starts at $C_g^{p_g}(p_\ell^H)$ from the highest level for loss probability p_ℓ^H moving from right to left. Notice that the orientation of $C_g^{p_g}(p_\ell^H)$ is equivalent to a clockwise rotation of $C_\ell^{p_\ell}(p_g^L)$ and a reversal of direction–according to the confidence operation $K^* = -K^T$. The shaded regions overlap the

Figure 1: Confidence Trajectory From Loss Over Gain Domain

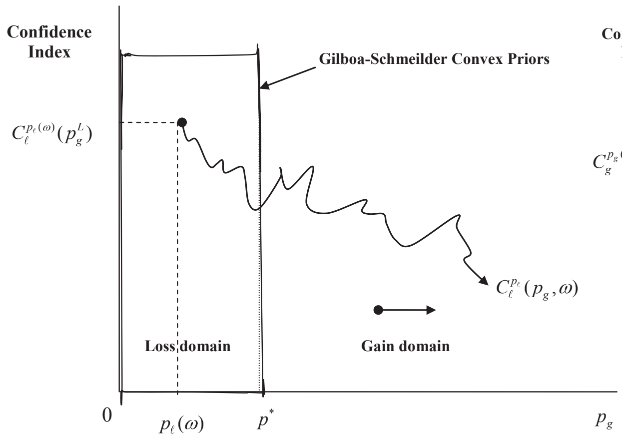
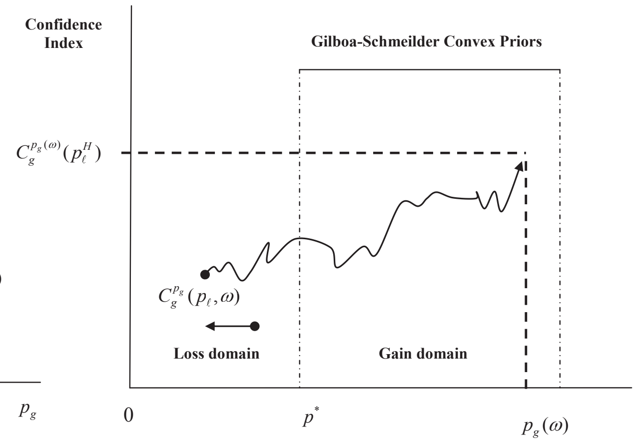


Figure 2: Confidence Trajectory From Gain Over Loss Domain



Gilboa and Schmeidler (1989) convex set of prior probabilities that determine the starting point for each trajectory. See also, (Feller, 1970, pp. 270-271). Moreover, notice the opposing pull or "force" in the loss direction. This is similar to the Langevin equation for Brownian motion of a particle with friction. It identifies loss aversion as t6he source of mean reversion in confidence and the popularity of Ornstein-Uhlenbeck processes in modeling behaviour in mathematical finance. See e.g. (Karatzas and Shreve, 1991, pg. 358).

Lemma 2.3 (Mean reversion in confidence)**.** *The source of mean reversion in sample paths for confidence is mean reversion.*

□

For the purpose of empirical exposition in this note, in the sequel we consider the strong but simple case where the kernel K is deterministic, π is Lebesgue measure, and $f(p_g) = 1$ and $h(p_\ell) = 1$ in (7) and (9).

2.3 Average confidence over Gilboa-Schmeilder convex priors

One immediate consequence of Theorem 2.2 is how to characterize expected confidence over a convex random set of priors. Relying on Jensen's Inequality, see (Feller, 1970, pp. 153-154), we begin with the familiar setup

$$E[C^{x(\omega)}(p)] = E[(Kf)(x(\omega))] = E[\int_{x(\omega)}^{y} K(x,z)f(z)dz] \tag{12}$$

$$K(x(\omega),z) = \int_{x(\omega)}^{z} (w(p)-p)dp = [W(z) - W(x(\omega))] - \frac{1}{2}[y^2 - x(\omega)^2] \tag{13}$$

Evaluation of $E[K(x(\omega),z]$ requires us to compute the following

$$E[W(x(\omega))] \geq W[E(x(\omega))] \tag{14}$$

by Jensen's Inequality for W convex in loss domain. Conversely, for W concave in gain domain

$$E[W(x(\omega))] \leq W[E(x(\omega))] \tag{15}$$

$$E[x(\omega)^2] \neq \Big(E[x(\omega)]\Big)^2 \tag{16}$$

Equations (14) and (15) plainly show that the "source function" W retains its general inverted S-shape popularized in the literature. However, the relationship in (16) indicates that equality between the kernel of the average, and the average of the kernel, does not hold in general. In fact, equality holds only if $x(\omega) = constant$. A result with probability zero in the context of Lebesgue measure, and our hypothesis of random convex priors. This implies

Lemma 2.4. *Let $K(x(\omega),y)$ be a random confidence kernel. Then*

$$E[K(x(\omega),y] \neq K(E[x(\omega)],y) \tag{17}$$

$\square$

Proposition 2.5 (Average across and within confidence levels)**.**
Let $[C^{x(\omega)}(p)] = [(Kf)(x(\omega))]$ *be the sample function of a random field of confidence with source* $x(\omega)$*. Let* $E[x(\omega)]$ *be the expected source. If* $E[C^{x(\omega)}(p)]$ *is average confidence path across confidence levels, and* $C^{E[x(\omega)]}(p)$ *is average confidence path generated from within Gilboa-Schmeilder source sets, then*

$$E[C^{x(\omega)}(p)] \neq C^{E[x(\omega)]}(p) \tag{18}$$

$\square$

Proof. Apply Lemma 2.4 to (12). $\square$

Corollary 2.6 (The case for conjugate priors: Approximate average confidence)**.**
If $E[x(\omega)^2]$ *is small, then* $E[C^{x(\omega)}(p)] \approx C^{E[x(\omega)]}(p)$*.*

$\square$

This implies that unless variance within and between source is small, an estimation strategy of using the average "source" to generate a sample function for a random field of confidence, will not yield good approximations to the average path for that sample function. Theoretically, this discrepancy can be reduced by the use of conjugate priors. See e.g. (DeGroot, 1970, pg. 159).

3 Applications

In this section we provide six applications for our operator. The first, explains the construction of our confidence field via a heuristic example. The second, is based on operations that transform Von Neuman Morgenstern (VNM) utility over loss/gain probability domains to characterize confident preferences. See (Von Neumann and Morgenstern, 1953, pg. 617). The third, is based on a simulation of our model to generate deterministic confidence paths for the identity function the confidence kernel domain. Fourth, we construct a trading algorithm motivated by the maxmin program implied by Gilboa and Schmeidler (1989). It provides a confidence based explanation for trading behavior of financial professionals reported in Abdellaoui et al. (2012). Fifth, we characterize the role of confidence in bubbles and crashes in large financial markets. Sixth, we tested our source set theory by estimating confidence betas across and within source sets derived from using CBOE VIX to split Gallup Economic Confidence Data into source sets.

3.1 Construction of confidence field

Let $\mathbf{e}_g \in \mathcal{D}(K)$ and $\boldsymbol{\epsilon}^\ell \in \mathcal{D}(K^*)$ be the vector valued identity function in the domain of K and K^*, respectively. In effect, $\mathbf{e}_g$ is a $r \times 1$ basis vector for gain domain with 1 in the g-th location and 0 otherwise, $g = 1, \ldots, r$. So that $I_r = [\mathbf{e}_1 \ldots \mathbf{e}_r]$ is a $r \times r$ identity matrix. Similarly, the derived basis for loss domain is defined

$$\boldsymbol{\epsilon}^i(\mathbf{e}_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad i = 1, \ldots, m \tag{19}$$

Thus we generate a *dual basis* for loss domains. So that $I^*_m = [\boldsymbol{\epsilon}^1 \ldots \boldsymbol{\epsilon}^m]$ is a $m \times m$ identity matrix. For example, let $\boldsymbol{z} = [z^1 \ldots z^r]^T$, where T stands for transpose, be a column vector that represents the coordinates of a vector valued function in $\mathcal{D}(K)$. We write

$$\boldsymbol{z} = z^1\mathbf{e}_1 + \ldots + z^r\mathbf{e}_r \tag{20}$$

$$= I_r\boldsymbol{z} \text{ (column notation) } = \boldsymbol{z}^T I_r \text{ (row notation)} \tag{21}$$

with respect to the basis in gain domain. By the same token, we expand the operator K in vector notation to get the matrix

$$K = [\boldsymbol{k}_{.1} \ldots \boldsymbol{k}_{.r}] \tag{22}$$

where $\boldsymbol{k}_{.j}, \;\; j = 1, \ldots, r$ is a $m \times 1$ column vector such that

$$\boldsymbol{k}_{.j}^T = [k_{1j} \; k_{2j} \; \ldots \; k_{mj}] \tag{23}$$

However, the i-th row of K is given by

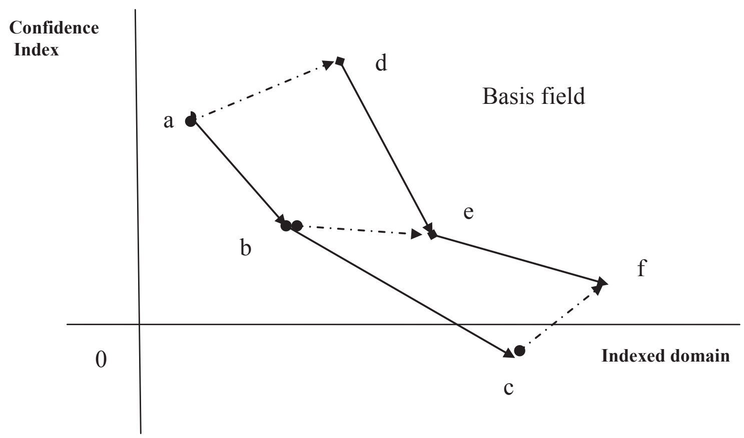
$$\boldsymbol{k}_{i.} = [k_{i1} \; k_{i2} \; \ldots \; k_{ir}], \;\; i = 1, \; \ldots, \; m \tag{24}$$

which runs through r-dimentional gain domain. Thus, the operation

$$C_\ell^\dagger = KI_r = [\boldsymbol{k}_{1.}^T \; \boldsymbol{k}_{2.}^T \; \ldots \; \boldsymbol{k}_{m.}^T]^T \tag{25}$$

generates m-rows of $1 \times r$ vectors. The ℓ-th row corresponds to the "projection" of initial loss probability p_ℓ over gain domain. It is a "basis field" for that initial loss probability, since it was generated by the identity matrix I_r in a manner consistent with the "resolution" of a vector in (21). A similar analysis shows that $C_g^{*\dagger} = K^* I_m^*$ generates r-rows of $1 \times m$ vectors. The g-th row corresponds to the "projection" of initial gain probability p_g over loss domain. It is a "basis field" for initial gain probability. In the context of the notation that follows, the "basis matrices" are $C_\ell^\dagger = [C_1^{p_1 T} \; \ldots \; C_m^{p_m T}]^T$ and $C_g^{*\dagger} = [C_1^{*p_1 T} \; \ldots \; C_r^{*p_r T}]^T$. In order to isolate the deterministic confidence "basis field" effect we did not randomize the matrices. For example, for loss priors that would require a process equivalent to $p_\ell(\omega) = p_\ell + \eta(\omega)$ where, for some variance σ^2, random draws are taken according to $\eta \sim (0, \sigma^2)$. Whereupon $C_\ell^\dagger(\omega)$ and $C_g^{*\dagger}(\omega)$ would be random matrices containing the *configurations* of sample functions of random fields of confidence generated by randomized priors. That analysis is outside the scope of this paper. Even so, the deterministic confidence fields capture the gist of the domain transformation(s) as indicated below. By way of illustration, consider the 2×3

Figure 3: Example of basis field orientation



matrix K, where $m = 2$ measures for loss and $r = 3$ measures for gain, correspond to a 6-point [indexed] probability domain such that

$$K = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad K^* = -K^T = \begin{bmatrix} -a & -d \\ -b & -e \\ -c & -f \end{bmatrix} \tag{26}$$

Figure 3 on page 16 depicts the orientation of the confidence basis field. There, we depict the two paths generated by K: $\overrightarrow{abc}$ and $\overrightarrow{def}$, as being downward sloping from left to right. These are the basis field generated over [indexed] gain domain (not shown on horizontal axis) for two prior loss probabilities p_{ℓ_1} and p_{ℓ_2}, say. They represent risk seeking over losses. By contrast, $K^* = -K^T$ generates three paths: $\overrightarrow{ad}$, $\overrightarrow{be}$, $\overrightarrow{cf}$. They are upward sloping from left to right. That orientation was obtained by reversing the direction of $\overrightarrow{abc}$ and $\overrightarrow{def}$ to $\overleftarrow{abc}$ and $\overleftarrow{def}$, and then rotating clockwise. These are the basis fields generated for three prior gain probabilities p_{g_1}, p_{g_2}, p_{g_3}, say. They represent "preference reversal" from risk seeking. Vizly, risk aversion over gain domain.

3.2 Confidence preferences

In what follows we introduce a functional representation for confidence induced preferences. Suppose that P and Q are probability measures that belong the the space M of probability measures such that the confidence operator $K(P,Q)$ is meaningful on $M \times M$. That is, K is defined on subsets M_P and M_Q of M. Let μ be a decomposable measure on M, and $V(P)$ be an abstract utility function defined on P. Classic VNM utility posits that if P is a probability measure over a suitable space X, then

$$U(P) = \int_{x \in X} u(x) dP(x) \tag{27}$$

However, in the context of our theory

$$V^P(Q) = (KU)(P) = \int_{M_Q} K(P,Q)U(Q)\mu(dQ) \tag{28}$$

$$= \int_{M_Q} K(P,Q)\left(\int_{x \in X} u(x)dQ(x)\right)\mu(dQ) \tag{29}$$

The nature of the product measure for $P \times Q \in M \times M$ and Fubini's Theorem, see (Gikhman and Skorokhod, 1969, pg. 97), allows us to write the foregoing as

$$V^P(Q) = \int_{M_Q} \int_{x \in X} K(P,Q(x))u(x)Q(dx)\mu(dQ) \tag{30}$$

Thus $V^P(Q)$ is the confidence adjusted VNM utility function. Our theory suggests that if P is in loss probability domain, and Q is in gain probability domain, then our subject is risk seeking over losses in hope of gain. Thus, $\mu = \mu^+$ is the hope of gain measure. In which case, the confidence adjusted VNM utility function $V^P(Q)$ is convex for given P. And we should rewrite (30) as:

$$V^P(Q) = \int_{M_Q} \int_{x \in X} K^*(P, Q(x)) u(x) Q(dx) \mu^+(dQ) \tag{31}$$

Recall that K^* is the adjoint of K so it induces the measure $\mu = \mu^-$. So by the same token, we can write

$$V^Q(P) = \int_{M_P} \int_{x \in X} K(P(x), Q) u(x) P(dx) \mu^-(dP) \tag{32}$$

According to our theory, because Q corresponds to gain probability, our subject is risk averse for given Q in fear of loss. Thus, the induced fear of loss measure is μ^-, and $V^Q(P)$ is concave for given Q. It is in effect an affine transformation of VNM. Equation 31 and (32) suggest that μ has a classic Hahn decomposition consistent with a Radon measure on M. See (Gikhman and Skorokhod, 1969, pg. 47) and (Edwards, 1965, pg. 178). In which case we just proved the following

Proposition 3.1 (Confidence induced decomposition of measures)**.** *The confidence operations K and K^* induce a decomposable measure on probability domain M.*

□

The confidence operations above transform VNM utility into (Tversky and Kahneman, 1992, pg. 303) value functions, and operationalize Yaari (1987) duality theory. In fact, following Tversky and Kahneman, let $f = (x_i, A_i),\ \ i = 1, \ldots, n$ be a prospect over disjoint events A_i with probability distribution characterized by $p(A_i) = p_i$. So (x_i, p_i) is a simple lottery. (Tversky and Kahneman, 1992, pg. 301) proposed the following scheme for decision weights (π_i) derived from operations on a probability weighting function w decomposed over gains w^+ and losses w^-. Rank x_i in increasing order so that it is dichotomized by a reference value. Positive outcomes are associated with $+$, and negative outcomes by $-$. Neutral outcomes by 0. So that for a given value function v over X we have

$$\pi_n^+ = w^+(p_n), \quad \pi_{-m}^- = w^-(p_{-m}) \tag{33}$$

$$\pi_i^+ = w^+\left(\sum_{s=i}^{n} p_s\right) - w^+\left(\sum_{s=i+1}^{n} p_s\right), \quad 0 \leq i \leq n-1 \tag{34}$$

$$\pi_i^- = w^-\left(\sum_{r=-m}^{i} p_r\right) - w^-\left(\sum_{r=-m}^{i-1} p_r\right), \quad -(m-1) \leq i \leq 0 \tag{35}$$

$$V(f^+) = \sum_{i=0}^{n} \pi_i^+ v(x_i) \tag{36}$$

$$V(f^-) = \sum_{i=-m}^{0} \pi_i^- v(x_i) \tag{37}$$

In the context of our model, let Π^- and Π^+ be the decision weights distribution for negative and positive outcomes, respectively. We claim that there exists some kernel $\tilde{K}(\Pi^-, \Pi^+)$ such that, assuming X is continuous, and using notation analogous to that for VNM utility

$$V^{\Pi^-}(\Pi^+) = V(f^+) = (\tilde{K}v)(f^+) = \int_{x \in X} \tilde{K}(\Pi^-, \Pi^+(x))v(x)\Pi^+(dx) \tag{38}$$

$$V^{\Pi^+}(\Pi^-) = V(f^-) = (\tilde{K}v)(f^-) = \int_{x \in X} \tilde{K}(\Pi^-(x), \Pi^+)v(x)\Pi^-(dx) \tag{39}$$

We summarize the above in the form of a

Lemma 3.2 (Confident decision operations)**.**

Let $f = (X, A)$ be a prospect with outcome space X and discrete partition A, and P be a probability distribution over X. Let $v : X \to \mathbb{R}$ be a real valued value function defined on X. Let Π^- and Π^+ be the distribution of decision weights over negative and positive outcomes, respectively, obtained by Tversky and Kahneman (1992) decision weighting operations. There *exists a confident decision operator $\tilde{K}$ defined on $\Pi^{-} \times \Pi^{+}$ such that the value functional over f is given by*

$$V(f) = (\tilde{K}v)(f) \tag{40}$$

$\square$

Remark 3.1. We note that Chateauneuf and Faro (2009) introduced a functional representation for confidence based on utility over acts–controlled by a confidence function defined on a level set of priors. Our functional is distinguished because it is predicated on decision weights, and the confidence operator introduced in this paper.

3.3 Model Simulation

To test the predictions of our theory, a sample of 30-probabilities were generated by separating the unit interval $[0, 1]$ into 29-evenly spaced subintervals. Including endpoints, we produce $n = 30$ observations. Thus, the fixed point probability $p^* = 0.34$ separated the interval into $m = [np^*] = 10$ loss probabilities, and $r = 20$ gain probabilities. So we were able to generate 20 confidence index measures for each loss probability p_ℓ by letting gain probabilities p_g run through gain domain. Similarly, we generated 10 confidence measures for each gain probability p_g by letting loss probabilities p_ℓ run through loss domain. This procedure generated a "term structure" for a confidence field as indicated in Figure 4 for K, and Figure 5 for K^* on page 22.

In Figure 4 on page 22, the highest curve, $Loss_1$, corresponds to the deterministic confidence trajectory $C^{p_\ell}_{\ell=1}(p_g)$. It represents the evolution of confidence when prior loss probability $p_{\ell=1} \in B_\ell$ is as close to zero as possible. In that case, our subject is overconfident when faced with small probability of loss–as indicated by the positive value of the index. As the prior probability of loss increases, confidence wanes. Thus, from $Gain_{11}$ (by abuse of notation this corresponds to the index $g = 11 \in \mathfrak{T}_g$) onwards our subject becomes *under* *confident and loss averse*. The lower confidence curves are based on higher initial values for prior loss probabilities. So if our subject begins the process with less confidence [or fear], then it carries over throughout the process. For example, at $Loss_{10}$, ie, $C^{p_\ell}_{\ell=10}(p_g)$ our subject begins with little or no confidence, ie, prior loss probability is $p_{\ell=10} \in B_\ell$, and becomes fearful of losing $Gain_4$ at index $g = 11$. By contrast, Figure 5 on page 22 depicts the transformation of loss domain into *hope of gain*. There, our subject is risk seeking over losses. In fact, for prior gain probability $p_{g=11} \in B_g$, when faced with the possibility of $Gain_{11}$, ie, $C^{*p_g}_{g=11}(p_\ell)$, onwards, our subject is overconfident over the entire loss domain indexed by $\mathfrak{T}_\ell$. So the curves in Figure 4 and Figure 5 indicate a "momentum factor" for confidence levels predicated on the convex set of prior probabilities B_ℓ and B_g. That is, for transformation matrix K, higher confidence induced by small *prior probability* of loss in B_ℓ serves as a confidence builder. This carries over deeper in gain domains before risk aversion kicks in and subjects become under confident and fearful. By the same token, for transformation matrix K^* we have overconfidence and hope induced by relatively small probability of gain in B_g. Therefore, the distribution of prior loss [gain] probabilities is a predictor of the evolution of confidence paths.

3.4 A confidence based program trading algorithm

In Figure 6 the convex region of feasible trades is enclosed by the dark red pseudo demand and supply lines and the x-axis. There, in the spirit of Gilboa and Schmeidler (1989) we can apply eight minimax, maximin, minimin, maximax criteria over the curves whose indexes' coincide with that of their respective generating prior(s). For example, for pseudo-demand

Figure 4: Confidence Motivated Psuedo Demand

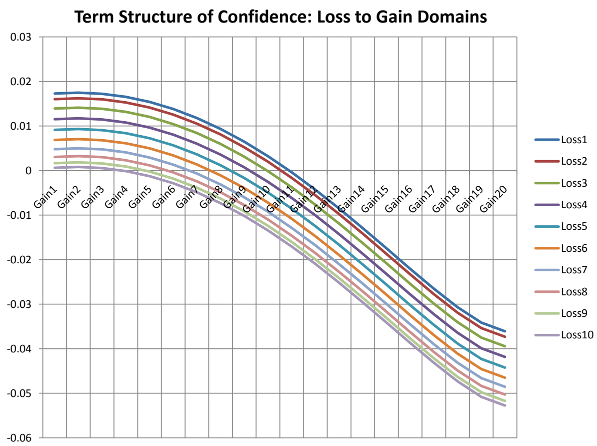


Figure 5: Confidence Motivated Psuedo Supply

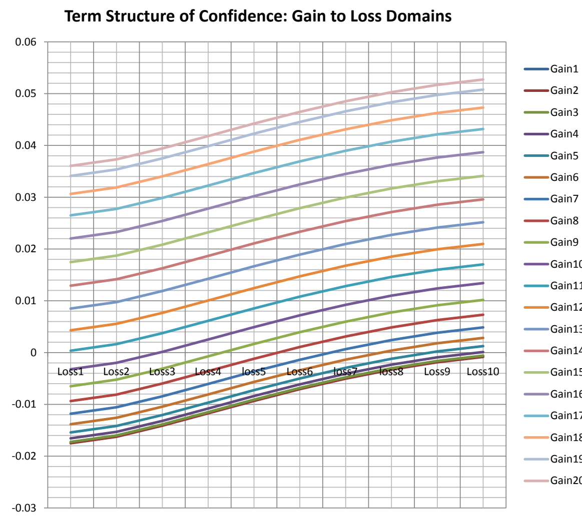


Figure 6: Confidence Motivated Psuedo Supply and Demand Equilibria

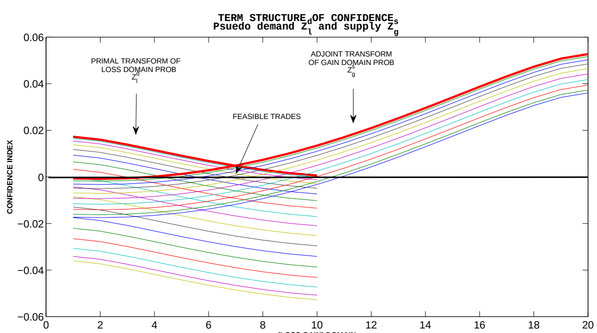
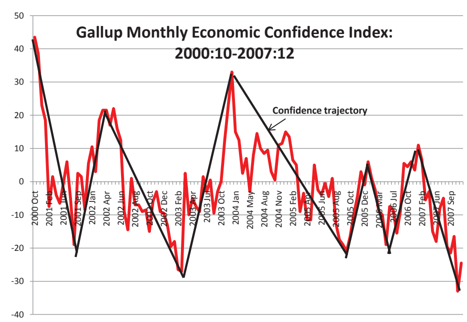


Figure 7: Gallup Monthly Economic Confidence Index from Survey Sampling



(d), and pseudo supply (s), we have

$$C_\ell^{\text{minimax, d}} = \min_\ell \max_g C_\ell^{p_\ell}(p_g) \tag{41}$$

$$C_\ell^{\text{maximin, d}} = \max_\ell \min_g C_\ell^{p_\ell}(p_g) \tag{42}$$

$$C_\ell^{\text{minimin, d}} = \min_\ell \min_g C_\ell^{p_\ell}(p_g) \tag{43}$$

$$C_\ell^{\text{maximax, d}} = \max_\ell \max_g C_\ell^{p_\ell}(p_g) \tag{44}$$

$$C_g^{\text{minimax, s}} = \min_g \max_\ell C_g^{*p_g}(p_\ell) \tag{45}$$

$$C_g^{\text{maximin, s}} = \max_g \min_\ell C_g^{*p_g}(p_\ell) \tag{46}$$

$$C_g^{\text{minimin, s}} = \min_g \min_\ell C_g^{*p_g}(p_\ell) \tag{47}$$

$$C_g^{\text{maximax, s}} = \max_g \max_\ell C_g^{*p_g}(p_\ell) \tag{48}$$

The intersection of these curves represent the feasible trading points. See e.g., the nodes in Figure 3 on page 16. For example a simple coherent program reads:

A CONFIDENCE BASED TRADE ALGORITHM

IF

$$\Big\{(C_\ell^{\text{minimax, d}} = C_g^{\text{minimax, s}}) \text{ or } (C_\ell^{\text{maximin, d}} = C_g^{\text{maximin, s}}) \text{ or}$$
$$(C_\ell^{\text{minimin, d}} = C_g^{\text{minimin, s}}) \text{ or } (C_\ell^{\text{maximax, d}} = C_g^{\text{maximax, s}})\Big\}$$

THEN $\Big\{$Confidence coherent and market equilibrium$\Big\}$

BEGIN

$\Big\{$Do not trade$\Big\}$

END

ELSE $\Big\{$Confidence incoherent$\Big\}$

BEGIN

IF $\Big\{$ no feasible trade$\Big\}$

THEN $\Big\{$stop$\Big\}$

ELSE $\Big\{$implement arbitrage strategy$\Big\}$

BEGIN

Let decompose source set $S := \bigcup_{i=1}^{N} A_i$

Let feasible arbitrage bound $:= \eta$, and $C_i \in A_i$

Let χ_A be an indicator, *select* $\bar{C} = \frac{1}{N} \sum_{i=1}^{N} C_i \chi_{A_i}(C_i)$

IF $|C_i \chi_{A_i}(C_i) - \bar{C}| > \eta$

THEN $\Big\{$trade accordingly$\Big\}$

END

END

Of necessity, there are $\binom{4}{1} \times \binom{4}{1} = 16$ possibilities to consider. Assuming that each of the 4 confidence coherent trades are feasible multiple equilibria, the other 12-possibilities suggest the existence or either arbitrage trading or no trade possibilities. (Hill, 2010, pg. 28) also presents a confidence based model different from ours in which an investor employs a program based on probability judgments. We note that even though our program implies the existence of incomplete markets, our "confidence coherence" approach is distinguished from Artzner et al. (1999) whose interest lie in coherent measures of risk. See also, Fedel et al. (2011).

Our model also has implications for asset pricing because it explains the trajectory and sensitivity of momentum strategies relative to prior probabilities, i.e. starting dates. For instance, Moskowitz et al. (2012) conducted a study in which they describe a purported "time series momentum" asset pricing anomaly as follows:

> [W]e find that the correlations of time series momentum strategies across asset classes are larger than the correlation of the asset classes themselves. This suggests a stronger common component to time series momentum across different assets than is present among the asset themselves. Such a correlation structure is not addressed by existing behavioral models.

Undeniably, Proposition 2.5 provides a "behavioral model[]" explanation for the seeming asset pricing anomaly. The "across asset class correlation" is our $E[C^{x(\omega)}(p)]$. It is tantamount to averaging across asset classes. By comparison, the "within asset class correlation" is our $C^{E[x(\omega)]}(p)$. In effect, Moskowitz et al. (2012) trading strategy is sensitive to "source sets". In the context of our model, their set of asset classes is a "source set" in which each class is accompanied by a different prior. Consequently, the within and across average is different. In fact, Maymin et al. (2011) conducted a study, and Monte Carlo experiments which plainly show that Moskowitz et al. (2012) momentum strategy is sensitive to start date, i.ee. priors.

3.5 Market Sentiment: hope, fear, bubbles, and crashes

If we think of our subjects as players in financial markets, then one admissible interpretation of Figure 6, is that when confidence levels are high, investors are not very risk averse. So they are on higher downward sloping [pseudo demand] confidence paths, i.e. demand for credit is high. [4]. Suppliers of credit for this veritable "irrational exuberance" are unable to satisfy demand on their current [pseudo supply] confidence path. So there is a structural shift to the left onto higher [pseudo supply] confidence curve, i.e. interest rates go up in response to increased demand for credit. It should be noted that the lowest level demand side confidence curves do not intersect with any upward sloping curves. That scenario represents credit rationing–investors whose confidence level is so low, and risk aversion is so high, that they opt out of the credit market because their demands are be met. Cf. Stiglitz and Weiss

[4]The analysis that follow is distinguished from Rigotti et al. (2008).

(1981). When there is a decrease in confidence, the confidence curves shift to the right. The foregoing scenario is reflected in Figure 7 which depicts Gallup Monthly Confidence Index data obtained from surveys for the period 2000-2007. That index is computed from the formula[5]:

$$\begin{aligned} &GDECIndex = \\ &\qquad \frac{1}{2} \times \Bigg(\%\text{Survey economic condition rated}\big[(Excellent\ +\ Good\big) -\ Poor)\big] \\ &\qquad + \%\text{Survey economic condition rated}\big(GettingBetter - GettingWorse\big)\Bigg) \end{aligned} \tag{49}$$

The index has a theoretical range $[-100,\ 100]$. There, positive numbers represent over confidence and negative numbers under confidence. The slope of the confidence trajectory over varying time ranges depict a term structure for confidence. In our model, the time index is replaced by indexed loss/gain probabilities. So we have a term structure for our confidence field. See Goldstein (2000). Perhaps most important, Gallup's index in (49) contain the component parts of Tversky and Wakker (1995) impact event which turns possibility into certainty; impossibility into possibility; versus a possibility more or less likely. Specifically, "Excellent + Good" implies "Certainty"; "Poor" implies "Impossibility"; "Getting Better" and "Getting Worse" are "possibility more or less likely" events. In effect, the GDECI is based on a subjective probability measure based on *bounded subadditivity*.

In practice, the scenario above takes place in a stochastic environment depicted by Figure 7. To see this analytically, let$M$ be the space of all probability measures; Ω be a sample space; $\mathfrak{S}$ and $\mathcal{F}$ the σ-field of Borel measureable subsets of $M$and Ω, resp. and π be a Levy-Prokhorov metric. See (Dudley, 2002, pg. 394). So that $(M \times M, \mathfrak{S} \times \mathfrak{S}, \pi)$ is

[5]See http://www.gallup.com/poll/123323/Understanding-Gallup-Economic-Measures.aspx. Last vitited 4/29/2012.

a Levy-Prokhorov measure space. The Levy-Prokhorov metric is a measure of the distance between two probability measures[6]. So it captures impact events of the type in (49). On some ambient space, assume that prior loss probabilities are randomized on some convex set $B \in \mathfrak{S}$, from loss to gain domains (and *vice versa* for gain to loss domain). This assumption is consistent with a subject's response to ambiguity aversion in the sense of Ellsberg (1961), and the proposed maximin program in Gilboa and Schmeidler (1989). Thus, we modify Definition 2.2 to account for a *generalized sample function* from the Markov random field of confidence, see Kinderman and Snell (1980) generated by prior beliefs. To that end, we have $p_g \in B_g$, and in particular $p_\ell(\omega) \equiv (p_\ell, \omega) \in \tilde{B}_\ell \times \Omega$. By abuse of notation, the sample function $C_\ell^{p_\ell}(p_g, \omega) \equiv C_\ell^{p_\ell(\omega)}(p_g)$ over gain probability domain is, in its most general form

$$\begin{aligned} C_\ell^{p_\ell}(p_g, \omega) &= \int_{p_\ell(\omega)}^{p_g \in B} K(p_\ell(\omega),\ p_g \in B_g)\mu(dp_g) \\ = \int_{p_\ell(\omega)}^{p_g \in B} w(p)\pi(dp) &- \delta\{\pi(p_\ell(\omega), p_g)|\ (p_\ell(\omega), p_g) \in \tilde{B}_\ell \times B_g\}, \quad \tilde{B}_\ell \times B_g \in \mathfrak{S}^2 \end{aligned} \tag{50}$$

where μ is a measure on gain [loss] probability domain, $\delta\{\pi(p_\ell(\omega), p_g)|\ \tilde{B}_\ell \times B_g\}$ is some function of the Levy-Prokhorov metric for loss-gain measures on $\tilde{B}_\ell \times B_g$. A similar relation holds for sample functions $C_g^{*p_g}(p_\ell, \omega)$ of random fields from gain to loss domains. The measureable Tversky and Wakker (1995) impact events in loss and gain domains are:

$$A_\ell = \{\omega|\ \omega \in p_\ell^{-1}(\tilde{B}_\ell) \subseteq \Omega\} \in \mathcal{F} \tag{51}$$

$$A_g = \{\omega|\ \omega \in p_g^{-1}(\tilde{B}_g) \subseteq \Omega\} \in \mathcal{F} \tag{52}$$

Consider an "animal spirit" $\omega \in \Omega$ in a large market of size $N = m + r$ with aggregate pseudo demand $D_{N,\ell}(K(\omega))$ and aggregate pseudo supply $S_{N,g}(K^*(\omega))$ to be defined below. According to our model, the elementary Tversky-Wakker event $\omega \in A_\ell \cap A_g$ is perceived

[6] A recent paper by Hill (2010) introduced a metric on probability spaces to characterize probability judgments. We leave the implementation of that for another day.

differently in loss and gain domains. If $p_\ell^j(\omega)$ is the random prior loss probability for the j-th subject, and $p_g^k(\omega)$ the random prior gain probability for the k-th [adjoint] subject, then we have

$$D_{N,\ell}(K(\omega)) = \sum_{j=1}^{m} C_\ell^{p_\ell^j}(p_g^j,\omega) \sim O_p^D(a(m)).\ \omega \in A_\ell \tag{53}$$

$$S_{N,g}(K^*(\omega)) = \sum_{k=1}^{r} C_g^{*p_g^k}(p_\ell^k,\omega) \sim O_p^S(b(r)),\ \omega \in A_g \tag{54}$$

where $O_p^D(a(m)),\ \ O_p^S(b(r))$ are probabilistic growth rates, for [slow varying] functions $a(m)$ and $b(r)$; $p_{(\cdot)}^i$ is i-th personal probability. If $D_{N,\ell}(K(\omega)) = S_{N,g}(K^*(\omega))$ in equilibrium, but $O_p^D(a(m)) \gg O_p^S(b(r))$, then aggregate demand is growing much faster than supply. So we have an eventual bubble. We summarize this in the following

Proposition 3.3 (Almost sure bubbles and crashes)**.** *Assume that confidence levels are ergodic. So that according to Birkhoff-Khinchin ergodic theorem there exist a limiting confidence trajectory $\hat{C}$. Let $D_{N,\ell}(K(\omega))$ and $S_{N,g}(K^*(\omega))$ be aggregate demand and supply for an elementary Tversky-Wakker impact event ω. Then the probability that there will be bubbles and crashes induced by confidence levels in excess of that consistent with market equilibrium, for some $\eta > 0$, is given by*

$$P\{\limsup_N |D_{N,\ell}(K(\omega)) - S_{N,g}(K^*(\omega))| > \eta\} = 1\ \ a.s \tag{55}$$

□

Proof. See Appendix B. □

Given sufficient time, another elementary Tversky-Waker elementary event or "animal spirit" $\omega_0 \in \Omega$ causes confidence to wane. So the [asymmetric] growth rates are reversed–demand growth is much lower than supply–and the market crashes to where $O_p^D(a(m)) \ll$

$O_p^S(b(r))$. The fluctuations of positive and negative growth rates, suggest that our relatively simple model is able to capture stylized facts about bubbles and crashes. Thus, confidence growth should be a policy control variable. In fact, consistent with our theory, the steep downward slope of the confidence trajectory which began in early 2007 in Figure 7 predicted the Great Recession which began around mid-2007.

3.6 Confidence source sets and VIX induced confidence beta

In this subsection we conduct a weak test of the source set hypothesis by using pseudo technical analysis of two popular time series for confidence: the Gallup Daily Economic Confidence Index (GDECI) and the Chicago Board of Exchange (CBOE) VIX daily series. There is solid theoretical and empirical reasons for the selection of these series evidence by Fox et al. (1996) (option traders exhibit bounded subadditivity) and Lemmon and Portniaguina (2006) (consumer confidence forecast small stocks but not variations in time series momentum). The GEDCI is computed from survey response as indicated by the formula in (49) which sounds like Tversky and Wakker (1995) impact event. VIX is computed from implied volatility for a sample of option prices, and it measures the market's expectation of near term volatility or uncertainty. See e.g. Chicago Board of Exchange (2009) for details on formula. Gallup did not report daily confidence index measures before 2008[7]. So our comparison with CBOE VIX daily series is limited to the post 2008 period between 2008:1:12–2012:03:28

Figure 8 on page 30 depicts the VIX daily close with basis field orientation for confidence paths. Undeniably, those paths mimic the predictions of our theory. We conducted an eyeball test by cross plotting GDECI vs. VIX and roughly identified two confidence regimes in the data. See Figure 9. We attribute those to Source Sets A and B which we extrapolated and plotted in Figure 10 and Figure 11. Source Set A represents the period between 20008:03:13–2009:09:17 which lies in the core of the global financial crisis, and Great Recession, in capital

[7]Private communication from Zach Bikus.

Figure 8: CBOE VIX Daily–Market Uncertainty

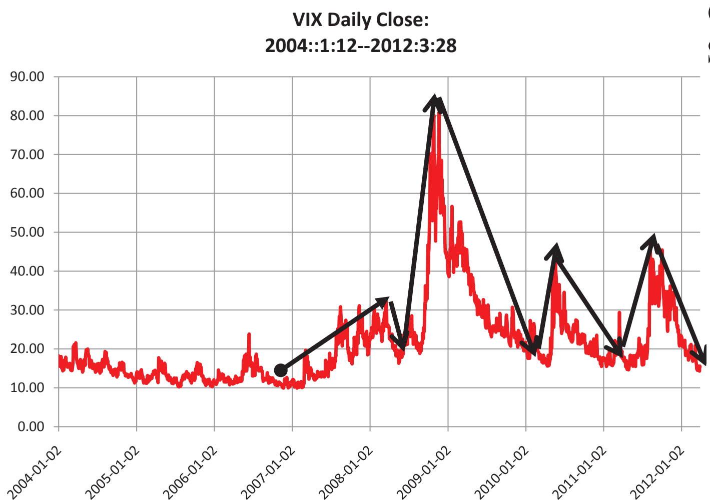


Figure 9: Gallup Daily Economic Confidence Index vs. VIX: Source Set $(A \cup B)$

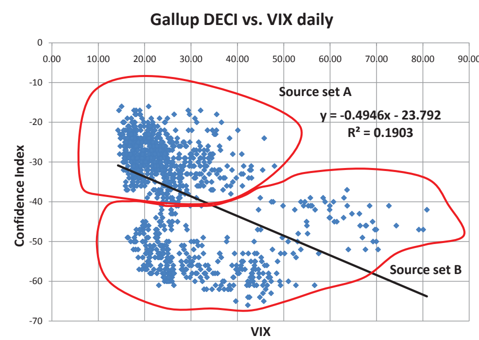


Figure 10: Gallup Daily Economic Confidence Index vs. VIX: Source Set A

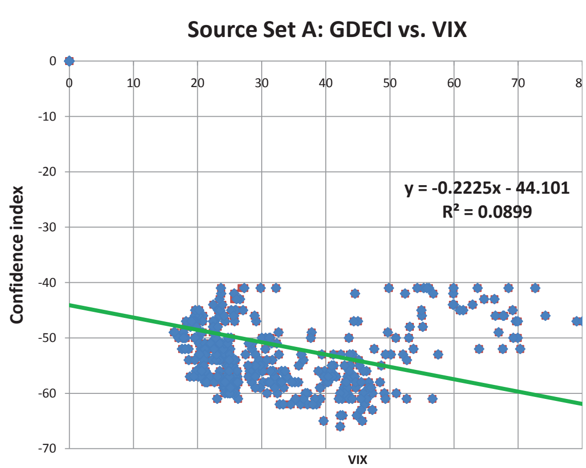
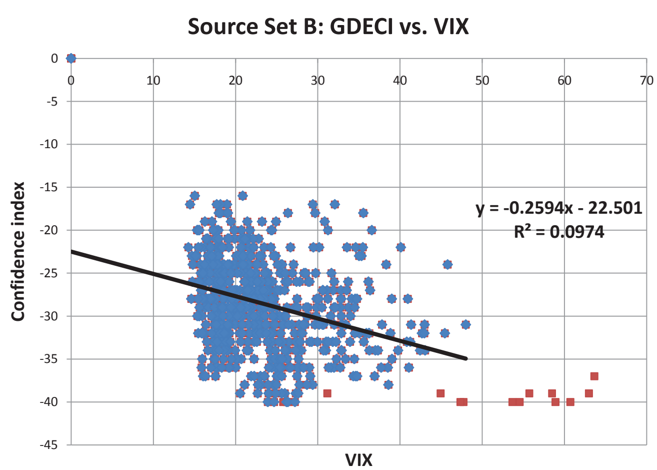


Figure 11: Gallup Daily Economic Confidence Index vs. VIX: Source Set B



markets. A period of great uncertainty or ambiguity in the context of our model. Source Set B represents the period 2009:09:18–2012:03:28. Evidently, agents in the economy were relatively less uncertain even though there was still ambiguity. We stated and tested the null hypothesis implied by Proposition 2.5 as follows:

HYPOTHESIS

$$\begin{aligned} &\boldsymbol{H_0}: \; E[C^{x(\omega)}(y) |\, x(\omega) \in \text{Source Set}(A \cup B)] \\ &= C^{E(x(\omega)}(y) |\, x(\omega) \in \{\text{Source Set}(A) \cup \text{Source Set}(B)\} \end{aligned} \tag{56}$$

$$\boldsymbol{H_a}: \; \boldsymbol{H_0} \text{ not true} \tag{57}$$

Here, $x(\omega)$ is a random "source" (in our case an unobserved prior probability induced by ambiguity) in the sets indicated for projection of confidence over y. We ran a simple linear regression of VIX on GEDCI (denoted as CONF) as a proxy for the average confidence path across Source Set $(A \cup B)$. In effect VIX is an instrumental variable for y. A similar regression was run within Source Set A and Source Set B[8]. To wit, $CONF^{(\text{source})}(VIX)$ is

[8]These erstwhile "source functions" are different from that in Abdellaoui et al. (2011). The latter is based on representation of probability weighting functions.

our instrument or measure for $C^{x(\omega)}(y)$. The results are reported as follows:

$$\text{CONF}_{A\cup B} = \underset{(0.9203)}{-23.7920} - \underset{(0.0324)}{0.4946}\,\text{VIX}_{A\cup B}, \quad R^2 = 0.1903,\ SSE = 142505.2,\ n_{A\cup B} = 1069 \tag{58}$$

$$\text{CONF}_{A} = \underset{(1.3104)}{-44.1011} - \underset{(0.0386)}{0.2225}\,\text{VIX}_{A}, \qquad R^2 = 0.0899,\ SSE = 43588.95,\ n_{A} = 383 \tag{59}$$

$$\text{CONF}_{B} = \underset{(0.3020)}{-22.5088} - \underset{(0.7212)}{0.2594}\,\text{VIX}_{B}, \qquad R^2 = 0.0974,\ SSE = 17307.76,\ n_{B} = 686 \tag{60}$$

The equations show that even though confidence beta and R^2 for Source Set A and Source Set B appear similar [9], Source Set A appears dissimilar to Source Set $(A \cup B)$. For example, approximately 9% of the variability in confidence levels is explained by VIX within each of source sets A and B, according to (59) and (60). By contrast, 20% of the variability in confidence is explained by VIX across source sets $(A \cup B)$ in (58). Source Set A represents a period of comparative uncertainty or ambiguity evidenced by its relatively large intercept term $\alpha_A = -44.1011$ and higher dispersion SSE. Thus, reflecting a comparatively high prior probability of loss $x^A = p_\ell^A \gg x^B = p_\ell^B$ and diffusion of confidence–consistent with response to the financial crisis in 2008 reflected in Source Set A data for 20008:03:13–2009:09:17, and the market crash predictions of Proposition 3.3. In order to test H_0 in (56) we employ the Chow-Test, see Chow (1960), explained in (Kmenta, 1986, pg. 421), whose statistic is given by

$$\frac{\frac{(SSE_{A\cup B} - SSE_A - SSE_B)}{K}}{\frac{(SSE_A + SSE_B)}{n_A + n_B - 2K}} \sim F^{K,\, n_A + n_B - 2K} \tag{61}$$

[9] There is a subset of Source Set B, vizly the period 2008:1:2–2008:03:12, which generated the confidence beta pricing equation

$$\text{CONF}_{subsetB} = \underset{(7.3474)}{-16.278} - \underset{(0.2835)}{0.7011}\,\text{VIX}_{subsetB}, \quad R^2 = 0.1152,\ n = 49,\ F = 6.1173,\ SSE = 849.1184$$

That may explain the discrepancy between (60) and (59).

There, $K = 2$ is the number of parameters in each equation. The computed statistic is $F = 434.8265$ with $(2, 1065)$ degrees of freedom. Thus $p \ll 0.01$ and we reject H_0 on the grounds that the "sources" or priors in Source Set A and Source Set B are not drawn from the same distribution.

Another useful exercise is comparison of relative confidence betas:

$$\frac{\beta_A}{\beta_B} = 0.86 \quad \frac{\beta_A}{\beta_{A \cup B}} = 0.45 \quad \frac{\beta_B}{\beta_{A \cup B}} = 0.52 \tag{62}$$

Roughly, the relative confidence beta (0.86) for Source Set A and Source Set B is larger than the relative beta for each source set across Source Set $A \cup B$. The relative betas are an implicit comparison of the growth in confidence. Thus, subjects in Source Set A started with a much higher prior loss probability implied by $\mathrm{CONF}_A(VIX)$ in (59). Consequently, they were comparatively less risk seeking than subjects in Source Set B which supports a steeper slope β_B. Recall that the vertical intercept in the basis field example illustrated in Figure 3 on page 16, as well as Figure 4 on page 22, correspond to initial value for prior loss probability p_ℓ. By the same token, slope comparison show there is more risk seeking in the market reflected by $\beta_{A \cup B}$ supported by Source Set $A \cup B$, compared to β_A, β_B in Source Sets A and B. Even though subjects in Source Set B started with a prior loss probability close to the market as evidenced by the intercept terms in (60) and (58). Whether these relative confidence betas could explain so called beta arbitrage in asset pricing theory is left to be seen. See e.g. Frazzini and Pedersen (2010).

4 Conclusion

We introduced a confidence kernel operator which establish a nexus between the multiple prior, and source function paradigms in decision theory. Further, the operator generates a field of confidence paths that mimic popular confidence indexes. So our model extends the solution space for confidence to integral equations and operator theory. Preliminary research in progress suggests that heteroskedasticity correction models for volatility clustering in econometrics mimic inverse confidence operations. So the confidence kernel operator may provide a "new" mechanism for heteroskedasticity correction by virtue of its data transforming mechanism. Cf. (Kmenta, 1986, pg. 280). Thus, we are able to answer questions like what preference functions in the domain of confidence kernels generate an observed confidence path. Additional research questions include but is not limited to whether our model can explain "butterfly effects" in confidence arising from small perturbation of prior probabilities.

A Proof of Theorem 2.2

This proof extends (Gikhman and Skorokhod, 1969, Thm. 2, pg. 189) to account for the probabilistic nature of the random domain $I(\omega) \subseteq M$. Let λ_n and $\phi_n(p)$ be the n-th eigenvalue and corresponding eigenfunction of $\Sigma(p_1, p_2)$. By definition $\Sigma(\cdot)$ is positive definite. Without loss of generality let $\pi(dp)$ be Lebesgue measure on M. According to Mercer's Theorem, see (Reisz and Sz.-Nagy, 1956, pg. 245) and (Loève, 1978, pg. 144) we can write

$$\Sigma(p_1, p_2) = \sum_{n=1}^{\infty} \lambda_n \phi_n(p_1) \bar{\phi}_n(p_2), \quad \lambda_n > 0 \; \forall n \tag{63}$$

$$\lambda_n \phi_n(p_1) = E\Big[\int_{I(\omega)} \Sigma(p_1, y)\phi_n(y)dy \Big| \mathcal{F}\Big], \quad E\Big[\int_{I(\omega)} \phi_m(p)\bar{\phi}_n(p)dp \Big| \mathcal{F}\Big] = \delta_{m,n} \tag{64}$$

where $I(\omega)$ is the random domain of definition for ϕ_n, and $\delta_{m,n}$ is Kronecker's delta. Define

$$\xi_n(\omega) = \int_{I(\omega)} C(p,\omega)\bar{\phi}_n(p)dp \tag{65}$$

$$E[\xi_n(\omega)\bar{\xi}_m(\omega)] = E[\xi_n(\omega)\bar{\xi}_m(\omega)]|\, \mathcal{F}] \tag{66}$$

$$= E\Big[\int_{I(\omega)}\int_{I(\omega)} \Sigma(p_1,p_2,\omega)\phi_n(p_1,\omega)\bar{\phi}_m(p_2,\omega)dp_1dp_2\Big]\Big|\mathcal{F}\ \Big] = \lambda_n\delta_{m,n} \tag{67}$$

$$C(p,\omega)\xi_n(\omega) = \int_{I(\omega)} \Sigma(p,y,)\phi_n(y)dy = \lambda_n(\omega)\phi_n(p,\omega);\ a.s.\ P \tag{68}$$

The latter representation introduces a probabilistic component to the proof which necessitated the ensuing modification. Consider the following probabilistic expansion based on the component parts above

$$E\Big[\Big|C(p,\omega) - \sum_{n=1}^{N}\xi_n(\omega)\phi_n(p,\omega)\Big|^2\ \Big|\mathcal{F}\ \Big] \tag{69}$$

$$= E\Big[\Sigma(p,p) - 2\sum_{n=1}^{N} C(p,\omega)\xi_n(\omega)\bar{\phi}_n(p,\omega) + \sum_{n=1}^{N}\lambda_n(\omega)|\phi_n(p,\omega)|^2\Big] \tag{70}$$

$$= E\Big[\Sigma(p,p) - \sum_{n=1}^{N}\lambda_n(\omega)|\phi_n(p,\omega)|^2\Big] \tag{71}$$

Let $\chi_{A'_N}$ be the characteristic function of an event A', and for $\epsilon_N(\omega) > 0$ sufficiently large define the random set

$$A'_N = \Big\{\omega\Big|\ \Big|\Sigma(p,p) - \sum_{n=1}^{N}\lambda_n(\omega)|\phi_n(p,\omega)|^2\Big| \geq \epsilon_N(\omega)\Big\}, \quad E[\chi_{A'_N}] = P(A'_N) \tag{72}$$

According to Parseval's Identity in L^2, see e.g. (Yosida, 1960, pg. 91), we have

$$\|\Sigma(p,p)\|^2 = \lim_{N\to\infty} E\Big[\sum_{n=1}^{N} \lambda_n(\omega)|\phi_n(p,\omega)|^2\Big] \tag{73}$$

$$\text{Let } \ \epsilon_N(\omega) = \sum_{j=N+1}^{\infty} \lambda_j(\omega)|\phi_n(p,\omega)|^2, \ \text{ so that } \ \epsilon_N(\omega) \downarrow 0 \tag{74}$$

Without loss of generality, assign $E[\xi_n(\omega)] = 0$. So that $\{\xi_n(\omega)\}_{n=1}^{\infty}$ is an *iid* mean zero sequence with finite second moments. That facilitates application of our results. By construction $C(p,\omega) \in C[0,1]$ and Σ is bounded and continuous in p. By virtue of the classic sup-norm $\|f\| = \sup_x |f(x)|, \ \ f \in C[0,1]$, and the induced metric $\varrho_C(f,g) = \sup_x |f(x) - g(x)|$ on $C[0,1]$, according to Theorem 1 in (Gikhman and Skorokhod, 1969, pp. 449-450), we have

$$\lim_{N\to\infty} \sup_{N} \sup_{|p-\bar{p}|\to 0} P\Big\{\Big|\Sigma(p,p) - \sum_{n=1}^{N} \lambda_n(\omega)|\phi_n(\bar{p},\omega)|^2\Big| > \epsilon_N(\omega)\Big\} = 0 \tag{75}$$

Thus we get a weaker result $\Sigma(p,p) \in L(M,\mathfrak{S},\pi)$. By absolute continuity of $\Sigma(p,p)$ on L, we apply Kolmogorov's Inequality in $L^2 \subset L$, and the probabilistic continuity criterion above to get

$$\lim_{N\to\infty} P(A'_N) = \frac{\lim_{N\to\infty} E\Big\{\Big|\Sigma(p,p) - \sum_{n=1}^{N} \lambda_n(\omega)|\phi_n(p,\omega)|^2\Big|^2 > \epsilon_N(\omega)\Big\}}{\epsilon_N^2(\omega)} = 0 \tag{76}$$

$$\text{So that } \ \lim_{N\to\infty} \sum_{n=1}^{N} E[\lambda_n(\omega)|\phi_n(p,\omega)|^2|\ \mathcal{F}] = \Sigma(p,p) \ \ a.s.\ P \tag{77}$$

By definition of $P(A')$ in (72) and the result in (77), all of which is based on the incipient expansion in (69), we retrieve the desired result

$$C(p,\omega) = \lim_{N\to\infty} \sum_{n=1}^{N} \xi_n(\omega)\phi_n(p,\omega) \ \ a.s.\ P \tag{78}$$

□

B Proof of Proposition 3.3

For notational convenience let

$$a(N) = \max\{O_p^D(a(m)), O_p^S(b(r))\} \tag{79}$$

$$C_\ell^i(\omega) = C_\ell^{p_\ell^i}(p_g^i, \omega) \tag{80}$$

$$C_g^{*i}(\omega) = C_g^{*p_g^i}(p_\ell^i, \omega) \tag{81}$$

Now expand the summands to account for any "surplus" demand or supply, and to account fo the fact that C can take positive or negative values.

$$\begin{aligned} a(N)^{-1}\Big|D_{N,\ell}(K(\omega)) - S_{N,g}(K^*(\omega))\Big| &= a(N)^{-1}\Bigg|\sum_{i=1}^{\min(m,r)} (C_\ell^i - C_g^i) \\ &+ \sum_{s=\min(m,r)+1}^{\max(m,r)} \left(\max\Big\{0, \frac{|C_\ell^s(\omega)| + C_\ell^s(\omega)}{2}\Big\} + \min\Big\{0, \frac{|C_\ell^s(\omega)| - C_\ell^s(\omega)}{2}\Big\}\right) \\ &+ \sum_{u=\min(m,r)+1}^{\max(m,r)} \left(\max\Big\{0, \frac{|C_g^{*u}(\omega)| + C_g^{*u}(\omega)}{2}\Big\} + \min\Big\{0, \frac{|C_g^{*u}(\omega)| - C_g^{*u}(\omega)}{2}\Big\}\right)\Bigg| \end{aligned} \tag{82}$$

$$\begin{aligned} &\leq a(N)^{-1} \sum_{i=1}^{\min(m,r)} |C_\ell^i - C_g^{*i}| + \\ &+ \sum_{s=\min(m,r)+1}^{\max(m,r)} \left|\left(\max\Big\{0, \frac{|C_\ell^s(\omega)| + C_\ell^s(\omega)}{2}\Big\} + \min\Big\{0, \frac{|C_\ell^s(\omega)| - C_\ell^s(\omega)}{2}\Big\}\right)\right| \\ &+ \sum_{u=\min(m,r)+1}^{\max(m,r)} \left|\left(\max\Big\{0, \frac{|C_g^{*u}(\omega)| + C_g^{*u}(\omega)}{2}\Big\} + \min\Big\{0, \frac{|C_g^{*u}(\omega)| - C_g^{*u}(\omega)}{2}\Big\}\right)\right| \end{aligned} \tag{83}$$

Let $I_1(N)$, $I_2(N)$, $I_3(N)$ be the value of the summands above in order. Assuming that supply and demand are satisfied where the curves intersect ie, $0 \leq |C_\ell^i - C_g^{*i}| \leq H$, then according to (79), the growth of $a(N)$ exceeds that of the summand for $I_1(N)$ when $|C_\ell^i - C_g^{*i}| = 0$ for countably many i. So we have

$$\lim_{N\to\infty} a(N)^{-1} I_1(N) = 0 \tag{84}$$

$$\lim_{N\to\infty} a(N)^{-1}\{I_2(N) + I_3(N)\} = c_0 > 0 \tag{85}$$

According to ergodic theory, see e.g. Birkhoff-Khinchine Theorem, (Gikhman and Skorokhod, 1969, pg. 127), the last equation implies that either $a(N)^{-1}I_2(N) = \hat{C}$ and $a(N)^{-1}I_3(N) = 0$ or *vice versa* where $\hat{C}$ is the limiting confidence trajectory. In which case we have from Chebychev's inequality

$$\mathrm{P}\limsup_{N\to\infty}\Big\{a(N)^{-1}|D_{N,\ell}(K(\omega)) - S_{N,g}(K^*(\omega))| > \eta\Big\} \tag{86}$$

$$\leq \frac{E\Big[a(N)^{-1}\big|D_{N,\ell}(K(\omega)) - S_{N,g}(K^*(\omega))\big|^2\Big]}{\eta^2} \tag{87}$$

$$= \frac{E[\hat{C}^2]}{\eta^2} \tag{88}$$

Since η is arbitrary, choose $\eta = |E[\hat{C}]|$ and the proof is done.

$\square$

Remark B.1. The proof also follows from application of Borel-Cantelli Lemma to the tail events described above.

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