Risk Management and Insurance Decisions under Ambiguity

Jimmy Martinez-Correa*

March 2012

ABSTRACT.

I study the impact of ambiguity on insurance decisions and the optimality of insurance contracts. My tractable approach allows me to study the interaction between risk and ambiguity attitudes. When insurance decisions are made independently of other assets, for a given increase in wealth, both risk and ambiguity attitudes interact in nontrivial ways to determine the change of coinsurance demand. I derive sufficient conditions to guarantee that the optimal coinsurance demand is decreasing in wealth. When a non-traded asset is introduced, my model predicts behavior that is inconsistent with the classical portfolio theory that assumes Subjective Expected Utility theory; however, it provides hints to a possible solution of the under-diversification puzzle of households. I also identify conditions under which more risk or ambiguity aversion decreases the demand for coinsurance. Additionally, I show a counterexample to a classical result in insurance economics where an insurance contract with straight deductible is dominated by a coinsurance contract. Finally, I find that a modified Borch rule characterizes the optimal insurance contract with bilateral risk and ambiguity attitudes and heterogeneity in beliefs.

*PhD Student, Department of Risk Management and Insurance, Georgia State University, J. Mack Robinson College of Business. e-mail: jimmigan@gmail.com. I am thankful for the helpful comments and constant support of my advisor Glenn Harrison and Ajay Subramanian. I also thank the participants of the Brown Bag seminar at the Department of Risk Management and Insurance at Georgia State University.

Table of Contents

Ellsberg [1961] studied the distinction between risk and ambiguity and its relevance for decision-making theory. He used a thought experiment to show that under certain situations many reasonable people tend, even after reflection, not to comply with the Savage [1972] postulates for subjective expected utility (SEU) theory.

Through his thought experiment, Ellsberg [1961] highlighted the importance of ambiguity and attitudes towards it. Camerer and Weber [1992] and Halevy [2007] are examples of studies that provide empirical evidence consistent with Ellsberg's findings.

I answer the following questions: What is the impact of ambiguity on the optimal coinsurance demand? Does this analysis change if the individual owns a non-tradable asset (e.g., Human capital)? What is the optimal insurance contract in the presence of ambiguity when both the insurer and insured might be ambiguity averse? Does this analysis have implications for portfolio theory?

A particular advantage of my approach is its tractability, which allows us to analyze the interaction between risk and ambiguity attitudes. However, this comes at the cost of generality because the approximation used to solve the problem is constructed to perform well in the *small*. Since Pratt [1964] the notion of small risks is relatively well understood. However, the concept of *small ambiguity* (or uncertainty)[1] is less straightforward[2] and is characterized by the convergence of the reminder in the approximation used here. My approach might be of special interest for experimental economics since the stakes and conditions in the laboratory are appropriate to generate small risks and/or uncertainties.

A clear prediction of my model is that, for a given increase in risk aversion, ambiguity averse individuals will increase their coinsurance demand less rapidly than ambiguity neutral agents. The intuition is that higher ambiguity aversion makes a marginal increase in insurance more valuable. Therefore, when risk aversion increases and more insurance is required to reduce variance, a smaller increase in insurance is needed because the additional coverage

[1] I use ambiguity and uncertainty as equivalent terms. However, there are important subtle differences between the two concepts but it is not the purpose of this study to describe the distinction between them. I follow the customary approach in the recent literature and use the term ambiguity (e.g., Gollier [2009].

[2] For a discussion on uncertainty in the small please see Maccheroni, Marinacci and Ruffino [2011b].

provides the double benefit of reducing risk and ambiguity.

Moreover, the response of the optimal coinsurance demand to changes in initial wealth will depend on the attitudes towards risk *and* ambiguity. For instance, given an increase in wealth, a subject with constant absolute risk aversion could still decrease her insurance demand if she exhibits decreasing absolute ambiguity aversion. I derive sufficient conditions to guarantee that the optimal coinsurance demand is decreasing in wealth.

A topic that is often overlooked in the analysis of insurance and portfolio choice is the presence of non-traded assets, such as Human capital. Cochrane [2007, p. 78] claims that:

> I have emphasized outside income [e.g., the return of human capital]..., even though it is rarely discussed in the modern portfolio theory literature. I think it's the most important and most overlooked component of portfolio theory, and that paying attention to it could change academic theory and the practice of the money management industry in important ways.

Mayers and Smith [1983] were the first to emphasize the importance of studying insurance decisions in the presence of traded and non-traded assets. Doherty and Schlesinger [1983] and Doherty [1984] studied a similar problem where the insurance demand can be affected by a background risk that is not insurable.

A non-traded asset introduces a new dimension to the insurance analysis under ambiguity. An individual is able to "*self-hedge*" if she can compensate high losses with high realizations of her non-traded human capital.

When ambiguity matters to a decision-maker and there exists a non-traded asset, an increase in risk aversion may or may not increase the demand for insurance depending on the incentives to "self hedge." This result is a counterexample to the Pratt-Arrow result which claims that higher risk aversion decreases the demand for the risky asset. This might happen when a marginal increase in insurance also increases the variance of wealth. Hence, an increment in risk aversion induces a reduction in the insurance demand. However, since an insurance decision problem can be interpreted as a portfolio problem, there is a much deeper result that has implications for portfolio theory and that drives this counterintuitive comparative static for risk aversion.

An important result is that, in the presence of ambiguity, the optimal risk-return allocation will not always be located in the efficient part of the classical mean-variance frontier. The reason is that high incentives to self-hedge create a trade-off between the risk and ambiguity dimensions of wealth that a decision maker must solve. Thus, an individual could appear to be "inefficient" from the perspective of classical portfolio analysis that assumes SEU utility maximization, although such action is optimal in the presence of ambiguity. I show that the counterintuitive comparative static for risk aversion that I described above can only arise when the optimal choice of an ambiguity averse individual lies in the "inefficient" part of the frontier.

A generalization of my model that includes traded assets, non-traded assets and insurance can be an alternative explanation for the under-diversification puzzle of households. The interaction of traded assets, non-traded assets and insurable assets, that can be each ambiguous or not, can result in observed behavior that is "under-diversified" from a classical perspective but optimal from a broader perspective. The main message is that deviations from traditional portfolio theory can be explained by expanding the concept of portfolio to include non-traded assets, to allow for preference representations that can explain attitudes towards ambiguity.

A simple extension of my framework that includes a risky asset, an ambiguous asset and risky income can rationalize "seemingly" irrational behavior. Massa and Simanov [2006] found that swedish investors tend to hold stocks that are positively correlated with their labor income, probably because these stocks are familiar to them, while wealthy individuals have a greater tendency to pick stocks that can help hedging their labor income risk. Standard explanations consider that wealthy investors are more sophisticated, while the observed positively correlated portfolios are usually explained by "behavioral biases" such as overconfidence, ignorance and familiarity. However, the simple model I propose suggests an alternative explanation. First, people exhibit ambiguity aversion towards less familiar stocks, which give them incentives to tilt their financial portfolio towards more familiar stocks, even if that sometimes means that the chosen portfolio is positively correlated with labor income. Second, an individual with sufficiently high initial wealth can overcome aversion towards ambiguity, just like a risk averter might diminish her aversion towards risk when wealth increases, and change his choices from portfolios tilted towards the risky asset to allocations that put more weight on the ambiguous asset.

In the last section I study the optimality of insurance contracts. First, I show that, in contrast to the traditional result in Arrow [1971], an insurance contract with a straight deductible may be dominated by an equivalent coinsurance contract if the decision-maker owns a non-traded asset. Second, a modified Borch rule characterizes the optimal contract when I allow for bilateral risk and ambiguity aversion and differences in beliefs. In particular, I provide conditions under which the coinsurance schedule with bilateral risk and ambiguity aversion is higher (or lower) than with bilateral ambiguity neutrality. When there is heterogeneity in beliefs no sharp predictions can be made.

To model individual preferences under ambiguity, I use the smooth ambiguity aversion model axiomatized by Klibanoff, Marinacci and Mukerji [2005], KMM henceforth.[3] They derive a preference representation that allows for a separation of the perception of ambiguity and attitudes towards it. Additionally, I adopt the quadratic approximation of the certainty equivalent of the KMM representation derived by Maccheroni, Marinacci and Ruffino [2011a]. This is an extension to the ambiguity domain of the Arrow-Pratt approximation, which allows for the model tractability I mentioned above. However, this may come at the cost of generality, since what is necessary and sufficient in the "small" may only be necessary in a more general setting (e.g., Eeckhoudt and Gollier [2000; p. 126]).

Some recent studies have examined the effect of ambiguity in insurance decisions and portfolio choices. Ju and Miao [2012] and Collard et al. [2009] study a dynamic infinite-horizon portfolio problem that allows for time-varying ambiguity and aversion towards it. They find, numerically, that ambiguity aversion increases the equity premium.

[3]The KMM model is susceptible to the ambiguity aversion paradoxes developed by Machina [2009] and Epstein [2010]; L'Haridon and Placido [2009] show data from an experiment supporting Machina's paradoxes.

However, Gollier [2009] found, in a static portfolio choice problem, that these numerical results rely on the particular calibration of the models. He identifies sufficient conditions under which more ambiguity aversion increases the demand for the ambiguous asset. Chebonnier and Gollier [2011] demonstrate that, in the KMM model, restrictions on risk and ambiguity attitudes are sufficient to guarantee that any uncertain situation that is undesirable at one wealth level is also undesirable at a lower wealth level. Nevertheless, they also show that one has to impose restrictions on both risk and ambiguity attitudes, as well as on the ambiguity structure, to guarantee that an increase in wealth will increase the demand of an ambiguous asset. Finally, Alary, Gollier and Treich [2010] and Snow [2011] study the effect of ambiguity on insurance decisions when they are made in *isolation*.

My tractable approach allows me to study in more depth the interactions between risk and ambiguity attitudes and to provide sharp predictions. Also, I consider a more general framework that models insurance decisions in the presence of other traded and non-traded assets, as well as preference representations that are sensitive both to risk and ambiguity.

Section 2 presents my approach to model decisions under ambiguity and analyzes the impact of ambiguity and attitudes towards it on the optimal coinsurance demand with and without a non-traded asset. Section 3 studies the optimality of an insurance contract with a straight deductible and the Borch rule when ambiguity matters to individuals. Section 4 concludes. Appendices show proofs of results and well-stablished results in the insurance literature when SEU is assumed.

1. Modeling Ambiguity and Attitudes towards Ambiguity

Consider a decision maker (DM) with initial wealth W_0 and random *end-of-period* wealth W. Suppose the agent exhibits aversion towards risk, which is captured by a utility function $u(.)$, with $u'(.) > 0$ and $u''(.) < 0$.

The DM perceives W as ambiguous. This implies that, instead of having a unique probability distribution for W, the subject behaves as if there is a bounded set of probability distributions, Δ, that are reasonably possible to her. Therefore, the DM is not able to commit to only use a particular distribution in this set. On the contrary, for a SEU maximizer, the set Δ will be a singleton.

Following KMM [2005], the individual has a subjective probability measure μ over Δ that captures the ambiguity perceived by her. According to KMM, the preferences of an individual that perceives wealth as ambiguous will have the following preference representation:

$$\int_{\Delta} \phi \left(\int_{S} u(W) dQ \right) d\mu, \tag{1}$$

where S is the state space of W, $Q \in \Delta$ is a probability measure over S, and ϕ is a map from reals to reals that captures attitudes towards ambiguity. If the subject perceives her future wealth as ambiguous, and is averse to this situation, ϕ will be concave in the same fashion that the preferences of a risk averse individual are represented by a concave utility function $u(.)$. An ambiguity loving (neutral) individual will have a convex (linear) ϕ, exactly parallel to the formal characterization of risk attitudes. I focus on the case of ambiguity aversion and risk aversion.

In the KMM framework, an increase in ambiguity can be characterized as an increase in the (subjective) variance of expected utility $\int_S u(W)dQ$, which is a random variable with subjective probability distribution μ. We can define a reduced compound probability distribution according to $\bar{Q} = \int_{\Delta} Q d\mu(Q)$. An individual that is ambiguity neutral, which is equivalent to behaving according to the postulates in Savage [1972], will maximize the SEU derived from $u(.)$ and $\bar{Q}$.

To make the models tractable, I use the second-order approximation of the CE of the KMM representation developed by Maccheroni, Marinacci and Ruffino [2011a], MMR hereafter. This is an extension to the ambiguity domain of the Arrow-Pratt approximation, where the risk and ambiguity premiums are characterized by the KMM preference representation. This CE is defined by the following equation:

$$\phi(u(\widetilde{CE})) = \int_{\Delta} \phi \left(\int_{S} u(W) dQ \right) d\mu \tag{2}$$

Define $W = W_0 + h$, where h represents the variable component of the end-of-period wealth. MMR approximate $\widetilde{CE}$ with the following quadratic smooth ambiguity functional that maps square integrable random variables to the reals:

$$\widetilde{CE} = W_0 + E_{\bar{Q}}[h] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(h) - \frac{\gamma}{2}\sigma^2_{\mu}(E[h]) + R_2(h) \tag{3}$$

where

$$\lim_{t\to 0} \frac{R_2(th)}{t^2} = 0. \tag{4}$$

The first two terms correspond to the Arrow-Pratt approximation of the CE of an individual that makes decisions according to the SEU defined by the compound probability distribution $\bar{Q}$ and utility function $u(.)$. The term $\theta = -\frac{u''(W_0)}{u'(W_0)} > 0$ is the local measure of absolute risk aversion. This reflects the risk aversion of an ambiguity neutral individual that behaves consistently with the SEU theory.

The new, third term is an ambiguity premium. The term $\gamma = u'(W_0)\{-\frac{\phi''(u(W_0))}{\phi'(u(W_0))}\} > 0$ captures the degree of ambiguity aversion (MMR [2011a; p. 6]). The term $\sigma^2_{\mu}(E[h])$ is the measure of ambiguity around the variable component of wealth perceived by the individual. In other words, it is the (subjective) variance of the expected value of h under probability measure μ. This expected value varies because the individual allows each probability distribution Q in Δ to be a possible candidate to estimate the expected value of h. Thus, in the finite case, there are n possible values that $E[h]$ can take, one for each probability distribution $Q_i \in \Delta$, $\forall i : 1, ..., n$. Finally, MMR [2011; p. 6] interpret $\sigma^2_{\mu}(E[h])$ as "model uncertainty" because it represents the possible deviations perceived by agents from a reference individual that maximizes SEU according to distribution $\bar{Q}$ and utility $u(.)$.

The convergence notion of the reminder of the approximation, as described in equation (4), defines the notion of "small" used here. The intuition is that the random component of wealth must be small enough such that the end-of-period wealth is not very different from initial wealth.

MMR [2011a; p. 9] decompose ambiguous random variables into three orthogonal components. They claim that for each $h \in L^2$, there exist unique $\bar{h} \in \mathbb{R}$, $h^* \in M$, with $E_{\bar{Q}}[h^*] = 0$ and $h^\perp \in M^\perp$, such that $h = \bar{h} + h^* + h^\perp$, where

$$M = \{h \in L^2 : \sigma_\mu^2(E[h]) = 0\}.$$

Define $\sigma_{\bar{Q}}^2(h) = \sigma_{\bar{Q}}^2(h^*) + \sigma_{\bar{Q}}^2(h^\perp)$ and $\sigma_\mu^2(E[h]) = \sigma_\mu^2(E[h^\perp])$. Further, $\bar{h}$ is the risk-free component of random variable h, h^* is a fair risky gamble (i.e., a gamble with zero expected payoff) and $h^\perp$ is its residual ambiguous component. MMR [2011a] show three possible configurations of a random variable. First, a gamble represented by random variable h is risk-free if and only if $h^* = h^\perp = 0$. Second, it is risky and unambiguous if and only if $h = \bar{h} + h^\perp$, because $\sigma_{\bar{Q}}^2(h) = \sigma_{\bar{Q}}^2(h^*)$ and $\sigma_\mu^2(E[h^\perp]) = 0$. Finally, the gamble is ambiguous if and only if h^* and $h^\perp$ different from zero.

There are several advantages of the approach to modelling ambiguity aversion used here. KMM [2005, p. 1868] claim that their representation allows for a separation of ambiguity and attitudes towards it, which provides a theoretical basis for undertaking comparative statics of ambiguity. Additionally, by adopting the MMR approximation, the KMM model becomes tractable and comparable to previous literature using the de Arrow-Pratt approximation, of course at the cost of generality that any approximation is exposed to.

2. Insurance Choices under Ambiguity

In this section I develop a model of coinsurance demand when the DM perceives the potential loss as ambiguous with and without a non-traded asset. I evaluate the robustness of some standard results in the insurance literature that assumes SEU, namely:[4]

1. Full coverage is optimal if coinsurance is available at a fair price. This is the Mossin Theorem developed in Mossin [1968]. Smith [1968] found similar results, thus I refer to this result as the Mossin-Smith Theorem.

[4] See Appendices B and C for the detailed derivation of the following results.

2. An increase in the degree of risk aversion will lead to an increase in the optimal demand for insurance at all levels of wealth. This result is a direct consequence of a standard result independently derived by Pratt [1964; p. 136] and Arrow [1963], thus I refer to it as the Pratt-Arrow result. They showed that an increase in absolute risk aversion decreases the demand for a risky asset. This would imply that a risk averse agent will demand more insurance to cover the loss than another individual that is less risk averse.[5]

3. If a subject exhibits decreasing absolute risk aversion (DARA), the optimal coinsurance demand decreases when initial wealth increases because her risk tolerance increases. Similar intuitive arguments can be made about CARA and IARA. However, Schlesinger [2000, p. 136] warns that "each of these conditions [DARA, CARA and IARA] is shown to be sufficient for the comparative-static effects[...], though not necessary."

4. The introduction of a non-traded asset, such as human capital, can significantly change some of the standard results. Mayers and Smith [1983] were the first to emphasize the importance of studying insurance decisions in the presence of traded and non-traded assets. Doherty [1984; p. 209] showed that the Mossin-Smith theorem only holds if the covariance between the non-traded asset and the insurable loss is negative. Following the terminology in Mayers and Smith [1983; p. 308], this covariance represents the individual's incentives to "*self-insure*". The sign of this covariance makes the insurance demand lower, equal or higher than in the absence of the non-traded asset. However, I prefer to use the term "*self-hedging*" to avoid confusion with the usage of "self-insurance" in the literature.[6] The intuition is that even if the insurance premium is fair, the DM might still not be willing to fully insure if she can compensate high

[5] In the context of insurance, the risky asset would be the retained loss. Thus, more insurance would translate into a lower exposure to the risky asset.

[6] Schlesinger [2000; p. 139] defines self-insurance as a mechanism that "lowers the financial severity of any loss that occurs".

losses with high realizations of her human capital. However, if the covariance is negative, the individual might want to fully insure if a health shock negatively affects her productivity, which would undermine her human capital. Finally, the introduction of a risky non-traded asset does not affect the Arrow-Pratt result.

I now turn to check the robustness of these results when ambiguity matters to individuals.

A. Model for Coinsurance Demand under Ambiguity

Consider a DM with initial wealth W_0 and exposed to a potential loss h that she perceives as ambiguous. Assume she is ambiguity averse. Thus, she behaves as if there is a bounded set Δ of possible probability distributions of h, with probability measure μ over that set. If the DM was ambiguity neutral, the compound probability distribution $\bar{Q}$ would be the only element in Δ. This is the case of a subject that complies with the postulates in Savage [1972] and behaves as if she maximizes the SEU defined by probability distribution $\bar{Q}$ and utility function $u(.)$. MMR's orthogonal decomposition of ambiguous acts implies that $h = E_{\bar{Q}}[h] + h^* + h^{\perp}$, with $h^* \in M$ and $h^{\perp} \in M^{\perp}$.

There is a risk neutral and ambiguity neutral insurer that is willing to offer the DM a coinsurance contract[7] that covers a fraction $\alpha \in (0, 1]$ of losses and in exchange for a premium π per unit of insurance. I assume that the insurer shares with the DM the same (compound) probability distribution $\bar{Q}$ of h and calculates the premium per unit of insurance according to $\pi = (1 + m)E_{\bar{Q}}[h]$, where $m \geq 0$ is the insurance loading that does not include any ambiguity charge. It is restrictive to assume that the insurer is both risk and ambiguity neutral and shares with the DM the compound distribution $\bar{Q}$. However, section 3 studies deviations from these restrictive assumptions.

If the DM buys insurance, her ambiguous end-of-period wealth is $W = W_0 - h - \alpha\pi + \alpha h$. I assume that the DM's preferences have a KMM representation with

[7] I focus in this section only on coinsurance contracts and deal later with the optimality of this type of contract and aversion towards ambiguity of the insurer.

concave functions $u(.)$ and $\phi(.)$ which, respectively, represent her aversion towards risk and ambiguity. Consequently, her maximization problem is:

$$\max_{\alpha} \int_{\Delta} \phi\left(\int_{S} u(W)dQ\right) d\mu \tag{5}$$

$$\text{subject to: } W = W_0 - h - \alpha\pi + \alpha h$$

Using the MMR quadratic approximation, this problem can be approximated by:

$$\max_{\alpha} E_{\bar{Q}}[W] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(W) - \frac{\gamma}{2}\sigma^2_{\mu}(E[W]) \tag{6}$$

$$\text{subject to: } W = W_0 - h - \alpha\pi + \alpha h$$

where θ and γ capture the degree of aversion towards risk and ambiguity, respectively. The definition of the end-of-period wealth, the insurance premium and the tripartite decomposition of h imply the following:

$$E_{\bar{Q}}[W] = W_0 - E_{\bar{Q}}[h] - \alpha(1+m)E_{\bar{Q}}[h] + \alpha E_{\bar{Q}}[h] = W_0 - (1+\alpha m)E_{\bar{Q}}[h]$$
$$\sigma^2_{\bar{Q}}(W) = (1-\alpha)^2\sigma^2_{\bar{Q}}(h) = (1-\alpha)^2[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})], \text{ and}$$
$$\sigma^2_{\bar{Q}}(E[W]) = (1-\alpha)^2\sigma^2_{\bar{Q}}(E[h^{\perp}])$$

Substituting the restriction into the objective function and differentiating with respect to α, the first order condition is:

$$\begin{aligned}[\alpha] : &- mE_{\bar{Q}}[h] - \frac{\theta}{2}\{2(1-\alpha)(-1)[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})]\} \\ &- \frac{\gamma}{2}\left\{2(1-\alpha)(-1)\sigma^2_{\mu}(E[h^{\perp}])\right\} = 0\end{aligned} \tag{7}$$

The optimal demand for coinsurance is:[8]

$$\alpha^{M1*}_{amb} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})] + \gamma\sigma^2_{\mu}(E[h^{\perp}])} \tag{8}$$

[8]The second order condition is satisfied.

To facilitate comparison, I assume that a DM that faces only risk (i.e., $\sigma^2_\mu(E[h^\perp]) = 0$), uses an objective distribution that is equal to the compound distribution $\bar{Q}$ to make choices. Under these conditions, the optimal insurance demand is defined by:

$$\alpha^{M1*}_{risk} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^\perp)]} \tag{9}$$

Alternatively, we can also interpret equation (9) in the light of a risk averse but ambiguity neutral individual. In the KMM framework, this agent is equivalent to a DM with the SEU preference representation given by utility $u(.)$ and reduced compound distribution $\bar{Q}$.

Comparative Statics of Risk and Ambiguity Attitudes

The Mossin-Smith theorem is robust to the introduction of ambiguity, because $\alpha^{M1*}_{amb} = 1$ if and only if $m = 0$. This can easily be seen from the definition of α^{M1*}_{amb}. Alary, Gollier and Treich [2010; p. 9] found a similar result.

Moreover, in the presence of ambiguity and ambiguity aversion, the direction of the Pratt-Arrow result is not affected, so an increase in risk aversion increases the insurance demand. Nevertheless, as we will see in the next section, this result may not hold in the presence of a non-traded asset. Assuming $m > 0$ such that $\alpha^{M1*}_{amb} \in (0, 1)$, the following derivative proves this statement:

$$\frac{\partial \alpha^{M1*}_{amb}}{\partial \theta} = \frac{mE_{\bar{Q}}[h]}{\{\theta\sigma^2_{\bar{Q}}(h) + \gamma\sigma^2_\mu(E[h^\perp])\}^2}[\sigma^2_{\bar{Q}}(h)] > 0 \tag{10}$$

with $\sigma^2_{\bar{Q}}(h) = \sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^\perp)$.

However, there is a second order difference with respect to the baseline model with only risk. The response of the insurance demand of an ambiguity averse agent to marginal increases in risk aversion is lower than that of an ambiguity neutral agent (i.e., $0 < \frac{\partial \alpha^{M1*}_{amb}}{\partial \theta} < \frac{\partial \alpha^{M1*}_{risk}}{\partial \theta}$), *ceteris paribus*. More generally, higher ambiguity aversion decreases the response of the insurance demand to higher risk aversion, i.e. $\frac{\partial^2 \alpha^{M1*}_{amb}}{\partial \theta \partial \gamma} < 0$. This implies that ambiguity averse individuals will increase insurance when risk aversion is higher, but at a slower rate than less ambiguity averse agents.

The intuition of this counterintuitive result is as follows. In the first order condition of the DM's maximization problem, the first term is the marginal cost of increasing insurance demand and the other two are the marginal benefits of reducing risk and ambiguity:

$$\underbrace{-mE_{\bar{Q}}[h]}_{\text{Marginal cost}} + \underbrace{\theta\{(1-\alpha)[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})]\}}_{\text{Risk reduction marginal benefit}} + \underbrace{\gamma\left\{(1-\alpha)\sigma^2_{\mu}(E[h^{\perp}])\right\}}_{\text{Ambiguity reduction marginal benefit}} = 0$$

Higher ambiguity aversion makes a marginal increase in insurance more valuable. Therefore, when risk aversion increases and more insurance is required to reduce variance, a smaller increase in insurance is needed when ambiguity aversion is higher because the additional coverage provides the double benefit of reducing risk and ambiguity.[9]

Moreover, given a level of risk aversion and subjective probability $\bar{Q}$, an ambiguity averse agent will demand more (co)insurance at the optimum than an ambiguity neutral individual (i.e., $\alpha^{M1*}_{amb} > \alpha^{M1*}_{risk}$). Alary, Gollier and Treich [2010; p. 12] showed that this is a general result for two states of nature. My tractable framework allows us to derive a "local" version of this result: *An increase in absolute ambiguity aversion increases the optimal insurance demand*. The following derivative proves this proposition:

$$\frac{\partial \alpha^{M1*}_{amb}}{\partial \gamma} = \frac{mE_{\bar{Q}}[h]}{\{\theta\sigma^2_{\bar{Q}}(h) + \gamma\sigma^2_{\mu}(E[h^{\perp}])\}^2}[\sigma^2_{\mu}(E[h^{\perp}])] > 0 \tag{11}$$

The Effects of Changes in Initial Wealth

It is usually assumed that people become less risk averse as they get wealthier. There are two important definitions of decreasing aversion in the economic literature. The first definition states that, in the portfolio choice problem with one safe and one risky assets, the

[9] This is a prediction that could be tested in the laboratory. If one is able to identify risk and ambiguity attitudes, then one could rank individuals according to risk aversion. The prediction above says that, an observed increase in risk aversion from θ_1 to θ_2 should increase more the insurance demand in the group of ambiguity neutral individuals than in the group of ambiguity averse agents.

optimal demand for the risky asset is increasing in wealth (Arrow [1963]; Pratt [1964; p. 136]). The second definition states that an individual has "decreasing aversion if any risk that is undesirable at some specific wealth level is also undesirable at all smaller wealth levels" (Cherbonnier and Gollier [2011; p. 1]). In the expected utility model both definitions of decreasing aversion are equivalent. A necessary and sufficient condition for this equivalency to hold is that the utility function u exhibits decreasing absolute risk aversion. One would expect that a similar condition would hold in the presence of ambiguity.

However, Chebonnier and Gollier [2011; p. 18] show, in a general setting of the KMM model, that the equivalency of the two definitions above does not hold in general. They demonstrate that restrictions on risk and ambiguity attitudes are sufficient to guarantee that any uncertain situation that is undesirable at one wealth level is also undesirable at a lower wealth level. Nevertheless, they show that one has to impose restrictions on both risk and ambiguity attitudes, *as well as* on the ambiguity structure, to guarantee that an increase in wealth will increase the demand of an ambiguous asset.

Since there is a close relationship between the portfolio choice and the coinsurance choice problems (see Schlesinger [2000; p. 135]), the results in Chebonnier and Gollier [2011] also apply to the choice of the optimal coinsurance rate.

I show below that in the "small," in the spirit of Pratt [1964] and as discussed by MMR [2011b], restrictions on risk and ambiguity attitudes are sufficient to guarantee that the optimal coinsurance demand is decreasing in wealth. This discrepancy between the small and the large is not uncommon.[10]

In the expected utility model, individual's risk preferences are represented by u and exhibit DARA if and only if $-u''(x)/u'(x)$ is decreasing in x. Similarly, ambiguity attitudes in the KMM model exhibit decreasing absolute ambiguity aversion (DAAA) if $-\phi''(z)/\phi'(z)$ is decreasing in z. The following definition characterizes this property for any function as

[10]For instance, in the classical expected utility model, "a necessary and sufficient condition for any pure small background risk to reduce the optimal exposure to other risks... is just necessary if one wants the comparative statics property to hold for any risk" (Eeckhoudt and Gollier [2000; p. 126]).

decreasing concavity.

Definition 1 (Cherbonnier and Gollier [2011; p. 3]): A function $f : \mathbb{R} \to \mathbb{R}$ satisfies (weak) Decreasing Concavity (DC) if $-f'/f''$ is non-increasing.

A fact that I will use in the proposition below is that $\sigma^2_{\bar{Q}}(h) > \sigma^2_{\mu}(E[h])$ (see Lemma 1 in Appendix A). This means that the subjective variance of $E[h]$, the measure of ambiguity in MMR [2010], is always going to be smaller than the subjective variance of h calculated with the compound distribution $\bar{Q}$.[11]

An important component of the KMM model is the function $v = \phi \circ u$. In fact, one of the key assumptions in KMM [2004; p. 1855] is that subjects behave according to SEU, derived from v and μ, when they are presented with second order acts.[12] We are now ready to state the first proposition.

Proposition 1: Assume $m > 0$. An increase in the initial wealth level W_0 reduces the optimal insurance demand if u and $v = \phi \circ u$ satisfy DC.

Proof. See Appendix A.

Proposition 1 implies that, in the "small," restrictions on risk and ambiguity attitudes of the KMM model are sufficient to guarantee that the optimal coinsurance demand under ambiguity is decreasing in wealth. No additional restrictions on the ambiguity structure are needed, like the ones imposed by Cherbonnier and Gollier [2011] in a general setting, to obtain the desired comparative static.[13]

[11] This is easily seen in a simple example. Suppose $h \in \{0, 100\}$, and the probability distribution could be either $\{.25, .75\}$ or $\{.75, .25\}$. Also assume that the subjective weights over these distributions are such that $\mu_1 = \mu_2 = .5$. Then, $\sigma^2_{\bar{Q}}(h) = 2500 > 625 = \sigma^2_{\mu}(E[h])$.

[12] The Ellsberg thought experiment can be represented by a second order urn that is ambiguous to subjects and that defines the possible configurations of the first order urn used to illustrate the Ellsberg paradox. A bet on the second order urn is a second order act.

[13] This result is potentially important for possible experimental applications attempting to test the predictions of models under ambiguity like the ones in this study. An experimenter trying to test such a model under ambiguity in the "small" will have to identify risk and ambiguity attitudes, as well as the perceived ambiguity, but does not have to control for the structure of ambiguity to test certain theoretical predictions.

B. Model for Coinsurance Demand in the Presence of Ambiguity and a Non-traded Asset

Suppose that the DM faces the same insurance decision as before, except that the individual owns a non-tradable and uninsurable asset (e.g., human capital) with risky return H that might be correlated with the loss. I argue that it could be reasonable to assume that subjects have a better idea about the uncertainty of their human capital, thus treating it as risk; meanwhile they may have *ambiguous* information about events that can negatively affect their health such as a genetic chronic disease. Although it is a fair question to ask if people perceive human capital as risky or ambiguous, this choice is more for expositional purposes since it allows us to study the potential trade-offs that agents might face in the presence of both risk and ambiguity.

As before, the risk and ambiguity neutral insurer offers coinsurance $\alpha \in (0, 1]$ and charges an insurance premium $\pi = (1 + m)E_{\bar{Q}}[h]$. Consequently, the ambiguous end-of-period wealth is $W = W_0 + H - h - \alpha\pi + \alpha h$. The MMR orthogonal decomposition implies that $h = E_{\bar{Q}}[h] + h^* + h^{\perp}$ and $H = E_{\bar{Q}}[H] + H^*$. Thus the maximization problem is:

$$\max_{\alpha} E_{\bar{Q}}[W] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(W) - \frac{\gamma}{2}\sigma^2_{\mu}(E[W]) \tag{12}$$

where

$$E_{\bar{Q}}[W] = W_0 + E_{\bar{Q}}[H] - (1 + \alpha m)E_{\bar{Q}}[h]$$

$$\sigma^2_{\bar{Q}}(W) = \sigma^2_{\bar{Q}}(H^*) + (1 - \alpha)^2[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})] - 2(1 - \alpha)cov_{\bar{Q}}(H^*, h^*)$$

$$\sigma^2_{\mu}(E[W]) = (1 - \alpha)^2\sigma^2_{\mu}(E[h^{\perp}])$$

The covariance between H and h is defined as $cov_{\bar{Q}}(H^*, h^*)$, because the ambiguous component of the loss ($h^{\perp}$) is orthogonal to the elements of set M (the set of risky and risk-free gambles). The joint probability distribution $\bar{Q}$ is the DM's best estimate to model the process driving the movements of H and h. The ambiguity of h implies that the DM behaves as if there were many joint distributions Q_i. The interpretation of this ambiguous situation is that the individual behaves as if he was sure about the marginal distribution in the direction of H, but is unsure about the marginal distribution of h. The distribution $\bar{Q}$ is the DM's best estimate of the joint distribution that is calculated by taking the average over the reasonably possible joint distributions with respect to a probability measure μ. The perceived ambiguity in the end-of-period wealth arises from $h^{\perp}$, and is captured by $\sigma_{\mu}^{2}(E[h^{\perp}])$.

Taking the derivative with respect to α, the first order condition is:

$$\begin{aligned}[\alpha]: &- mE_{\bar{Q}}[h] - \frac{\theta}{2}\{2(1-\alpha)(-1)[\sigma_{\bar{Q}}^{2}(h^{*}) + \sigma_{\bar{Q}}^{2}(h^{\perp})] - 2(-1)cov_{\bar{Q}}(H^{*}, h^{*})\} \\ &- \frac{\gamma}{2}\{2(1-\alpha)(-1)\sigma_{\mu}^{2}(E[h^{\perp}])\} = 0\end{aligned} \tag{13}$$

The optimal demand for insurance is:

$$\alpha_{amb}^{M2*} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta\sigma_{\bar{Q}}^{2}(h) + \gamma\sigma_{\mu}^{2}(E[h^{\perp}])} - \frac{cov_{\bar{Q}}(H^{*}, h^{*})}{\sigma_{\bar{Q}}^{2}(h) + \frac{\gamma}{\theta}\sigma_{\mu}^{2}(E[h^{\perp}])} \tag{14}$$

where $\sigma_{\bar{Q}}^{2}(h) = \sigma_{\bar{Q}}^{2}(h^{*}) + \sigma_{\bar{Q}}^{2}(h^{\perp})$.

For comparison purposes, an ambiguity neutral but risk averse DM that has a subjective distribution $\bar{Q}$, will exhibit an optimal insurance demand defined by:

$$\alpha_{risk}^{M2*} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta\sigma_{\bar{Q}}^{2}(h)} - \frac{cov_{\bar{Q}}(H^{*}, h^{*})}{\sigma_{\bar{Q}}^{2}(h)} \tag{15}$$

Comparative Statics of Risk and Ambiguity Attitudes

The optimality of full insurance is not qualitatively changed by the introduction of ambiguity and a non-traded asset. As shown by Doherty [1984], the Mossin-Smith theorem holds only if $cov_{\bar{Q}}(H^{*}, h^{*}) \leq 0$. However, there is a quantitative difference that I explain in the next paragraph.

Given the same level of risk aversion and subjective probability distribution $\bar{Q}$, an ambiguity averse agent will demand more (co)insurance at the optimum than an ambiguity neutral individual, because $\alpha_{amb}^{M2*} > \alpha_{risk}^{M2*}$. The inclusion of a non-traded asset does not change significantly a similar result shown by Alary, Gollier and Treich [2010; p. 12]. They show that, for two states of nature and given risk aversion, an ambiguity averse agent demands more insurance than an ambiguity neutral person when insurance decisions are modeled in isolation. However, depending on the sign and level of covariance, the optimal demand α_{amb}^{M2*} can be smaller, equal or greater than α_{amb}^{M1*} and/or α_{risk}^{M1*}. Finally, as the next proposition shows, my tractable model allows us to derive a "local" version of the comparative static of ambiguity attitudes.

Proposition 2: In the presence of ambiguity and aversion towards it, if the incentives to *self-hedge*, represented by $cov_{\bar{Q}}(H^*, h^*)$, are low (high) enough, more ambiguity aversion decreases (increases) the optimal demand for insurance.

Proof.

$$\begin{aligned}\frac{\partial \alpha_{amb}^{M2*}}{\partial \gamma} &= -\frac{mE_{\bar{Q}}[h][-\sigma_{\bar{Q}}^2(h^{\perp})]}{[\theta\sigma_{\bar{Q}}^2(h) + \gamma\sigma_{\mu}^2(E[h^{\perp}])]^2} - \left[-\frac{cov_{\bar{Q}}(H^*, h^*)\left[\frac{\sigma_{\mu}^2(E[h^{\perp}])}{\theta}\right]}{\sigma_{\bar{Q}}^2(h) + \frac{\gamma}{\theta}\sigma_{\mu}^2(E[h^{\perp}])]^2} \right] \\ &= \frac{[mE_{\bar{Q}}[h] + \theta cov_{\bar{Q}}(H^*, h^*)]\sigma_{\mu}^2(E[h^{\perp}])}{[\theta\sigma_{\bar{Q}}^2(h) + \gamma\sigma_{\mu}^2(E[h^{\perp}])]^2}\end{aligned}$$

Define a threshold covariance $\kappa_{H,h;\gamma}^{M2*} = -\frac{mE_{\bar{Q}}[h]}{\theta}$. Therefore,

$$\frac{\partial \alpha_{amb}^{M2*}}{\partial \gamma} = \begin{cases} < 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) < \kappa_{H,h;\gamma}^{M2*} \\ = 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) = \kappa_{H,h;\gamma}^{M2*} \\ > 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) > \kappa_{H,h;\gamma}^{M2*} \end{cases} \tag{16}$$

■

The optimal demand α_{amb}^{M2*} is always greater than one whenever $cov_{\bar{Q}}(H^*, h^*) < \kappa_{H,h;\gamma}^{M2*}$. If the supply of insurance is restricted to $[0, 1]$, then we would only observe that $\frac{\partial \alpha_{amb}^{M2*}}{\partial \gamma} \geq 0$. This is consistent with Alary, Gollier and Treich [2010] who found that the demand for insurance is increasing in ambiguity aversion when the insurance decisions are made in isolation. However, if the agent is allowed to own a non-traded asset, to have an insurance demand greater than one and $cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\gamma}$, she will reduce her demand for insurance when ambiguity aversion increases.

Gollier [2009] found that under certain conditions more ambiguity aversion can increase the demand for the ambiguous asset. We can infer that if my model is framed in a portfolio choice context without restrictions on the asset demand, the incentives to "*self-hedge*" may constitute a rationale different from Gollier [2009] to explain why more ambiguity aversion might increase the demand for the uncertain asset.

An important result is that the introduction of ambiguity and a non-traded asset may contradict the Pratt-Arrow result that insurance demand increases with risk aversion. I state the result in the following proposition and explain its rationale in the next section.

Proposition 3: When ambiguity matters to the individual, an increase in risk aversion may or may not increase the demand for insurance depending on the incentives to "*self hedge*" created by $cov(H^*, h^*)$.

Proof: The derivative of the optimal insurance demand with respect to the absolute risk aversion coefficient is given by:

$$\begin{aligned}\frac{\partial \alpha^{M2*}_{amb}}{\partial \theta} &= -\left[-\frac{mE_{\bar{Q}}[h] \times [\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})]}{[\theta[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})] + \gamma\sigma^2_{\mu}(E[h^{\perp}])]^2}\right] \\ &\quad - \left[-\frac{cov_{\bar{Q}}(H^*, h^*) \times \left[-\frac{\gamma \times \sigma^2_{\mu}(E[h^{\perp}])}{\theta^2}\right]}{[[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})] + \frac{\gamma}{\theta}\sigma^2_{\mu}(E[h^{\perp}])]^2}\right] \\ &= \frac{mE_{\bar{Q}}[h] \times [\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})] - \gamma cov_{\bar{Q}}(H^*, h^*)\sigma^2_{\mu}(E[h^{\perp}])}{[\theta[\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})] + \gamma\sigma^2_{\mu}(E[h^{\perp}])]^2}\end{aligned} \tag{17}$$

Define a threshold covariance $\kappa^{M2*}_{H,h;\theta} = \frac{mE_{\bar{Q}}[h]}{\gamma} \times \frac{\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})}{\sigma^2_{\mu}(E[h^{\perp}])}$. Therefore,

$$\frac{\partial \alpha^{M2*}_{amb}}{\partial \theta} = \begin{cases} > 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\theta} \\ = 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) = \kappa^{M2*}_{H,h;\theta} \\ < 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) > \kappa^{M2*}_{H,h;\theta} \end{cases} \tag{18}$$

■

In the limit, when the agent behaves as a SEU-maximizer, we obtain the Pratt-Arrow result because the optimal insurance demand will be non-decreasing in risk aversion. When the ambiguity averse agent converges to an SEU-maximizer ($\gamma \to 0$) or ambiguity disappears ($\sigma_{\mu}^2(E[h^{\perp}]) \to 0$), the threshold $\kappa_{H,h;\theta}^{M2*}$ will never be reached for variables with finite covariance. Thus, we would only observe that $\frac{\partial \alpha_{amb}^{M2*}}{\partial \theta} > 0$. However, proposition 3 shows an exception to Pratt-Arrow result that arises when ambiguity and a non-traded asset are present. The rationale of this counterintuitive result, which I explain in the next section, relies on possible trade-offs between risk and ambiguity that the DM must solve.

A Trade-off between Risk and Ambiguity: Are Subjects Behaving "Inefficiently"?

The Pratt-Arrow result with only risk, or with ambiguity but with a SEU-maximizer, implies that more risk aversion increases the insurance demand because the lower variance that higher risk aversion demands can always be met with more insurance. In Proposition 3 I showed that if ambiguity matters to the agent and there is a non-traded asset, more risk aversion does not always increase the insurance demand. The reason is that in this environment the lower variance of wealth required by an increase in risk aversion is not necessarily achieved through more insurance if the incentives to *self-hedge* are high enough.

The source for this counterintuitive result is that the DM might have to face a potential trade-off between risk and ambiguity that depends on the incentives to self-hedge. This can induce the subject to behave *as if* she was choosing voluntarily a risk allocation that is "inefficient" from a SEU perspective. However, this allocation might be optimal once ambiguity is taken into account. I explain below the origin of the risk-ambiguity trade-off by expressing the insurance problem in the form of a portfolio problem. This will allows me to draw some possible implications of my results to the theory of portfolio.

The insurance decision problem can be interpreted as a *portfolio problem* (Schlesinger [2000; p. 135]), since a higher (lower) demand for insurance is equivalent to a lower (higher) demand for the risky asset. Therefore, I can define a mean-variance frontier, in the same spirit as the financial literature, by finding the locus of $\{\sigma^2_{\bar{Q}}(W(\alpha)), E_Q[W(\alpha)]\}$ pairs generated by every value $\alpha \in [0,1]$. Define $\alpha^*_{MinVar} = 1 - \frac{cov(H^*,h^*)}{\sigma^2_{\bar{Q}}(h)}$ as the insurance level that provides the wealth with minimum variance.[14] Moreover, suppose that $\alpha^*_{MinVar} \in (0,1)$, which is guaranteed by $cov(H^*, h^*) \in (0, \sigma^2_{\bar{Q}}(h))$.

The efficient (inefficient) portion of the mean-variance frontier is usually defined as the set of mean-variance allocations that offer the highest (lowest) expected wealth at a given level of risk. In my framework, the efficient (inefficient) part of the frontier is defined as the set of $\{\sigma^2_{\bar{Q}}(W(\alpha)), E_Q[W(\alpha)]\}$ pairs where $\frac{\mathrm{d}\,\sigma^2_{\bar{Q}}(W(\alpha))}{\mathrm{d}\,E_{\bar{Q}}[W(\alpha)]} > (<)0$ (See Lemma 2 in the Appendix A).

Definition 2: The mean-variance allocation $\{\sigma^2_{\bar{Q}}(W(\alpha)), E_Q[W(\alpha)]\}$ will be located on the SEU-efficient (SEU-inefficient) part of the frontier whenever the insurance demand $\alpha < \alpha^*_{MinVar}$ ($\alpha > \alpha^*_{MinVar}$).

I refer to the efficient (inefficient) region as the *SEU-efficient* (*SEU-inefficient*) part of the frontier. This emphasizes the fact that a subject, who would appear inefficient in the light of SEU theory, might optimally choose such an allocation to deal both with risk and ambiguity. I show below that this is the case under certain conditions.

Proposition 4: Assume that the loss h is ambiguous and the non-traded asset H is risky. An ambiguity neutral agent will choose an optimal mean-variance allocation that will be located on the *SEU-efficient* region of the frontier.

This proposition is obvious since a subject that does not care about ambiguity and behaves according to Savage [1972], will choose an allocation that maximizes expected wealth given a certain level of variance.

Proof. According to definition 2, the optimal allocation $\{\sigma^2_{\bar{Q}}(W(\alpha^{M3*}_{risk})), E_{\bar{Q}}[W(\alpha^{M3*}_{risk})]\}$ induced by α^{M2*}_{risk} will be located on the efficient part of the frontier because:

$$\alpha^{M2*}_{risk} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta\sigma^2_{\bar{Q}}(h)} - \frac{cov_{\bar{Q}}(H^*,h^*)}{\sigma^2_{\bar{Q}}(h)} < 1 - \frac{cov(H^*,h^*)}{\sigma^2_{\bar{Q}}(h)} = \alpha^*_{MinVar}$$

[14] This is easily found by minimizing $\sigma^2_{\bar{Q}}(W)$ with respect to α.

■

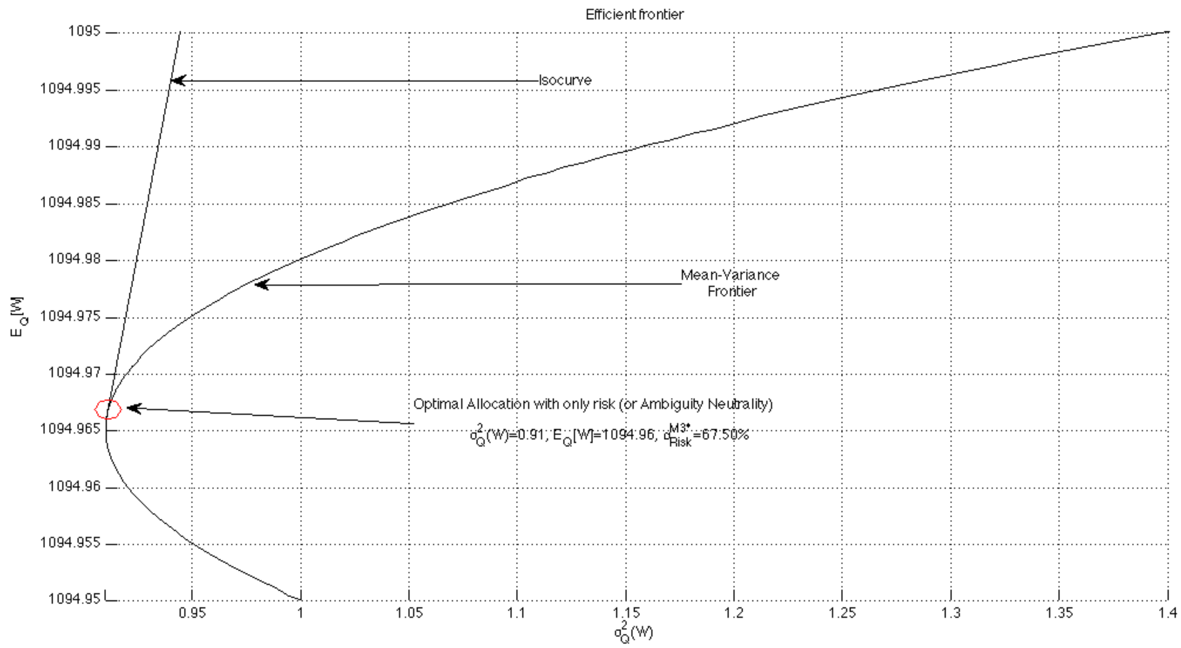

Figure 1: Portfolio Problem Interpretation of a Simple Insurance Decision. Parametric assumptions: $\theta = \gamma = 2, \sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp}) = \sigma^2_{\mu}(E[h^{\perp}]) = 1, W_0 = 1000, cov_{\bar{Q}}(H^*, h^*) = 0.3, m = .01, E_{\bar{Q}}[H] = 100$ and $E_{\bar{Q}}[h] = 5$. The isocurve contains the $\{\sigma^2_{\bar{Q}}(W), E_Q[W]\}$ pairs that make the individual indifferent to the optimal allocation which is determined by the tangency with the mean-variance frontier.

Figure 1 shows the optimal mean variance allocation induced by the optimal insurance demand of a SEU maximizer. In the particular parametrization used to construct the figure, this optimal demand is $\alpha^{M2*}_{risk} = 67.5\%$ (see the tangency of the isocurve and the mean-variance frontier). Remember that, according to definition 2, a mean-variance allocation will be SEU-efficient if the optimal insurance demand is less than the coinsurance demand that provides minimum variance, which is $\alpha^*_{MinVar} = 70\%$ in Figure 1. Although it is still possible in this example for the DM to choose an insurance coverage above α^*_{MinVar}, she will not do it so because there is still an attainable allocation (in the SEU-efficient part of the frontier) that provides higher expected return for the same variance. Finally, if $cov(H^*, h^*) < 0$ then $\alpha^*_{MinVar} = 100\%$, provided the insurer only offers coinsurance $\alpha \in (0, 1]$; as a consequence all mean-variance allocations induced by $\alpha \in (0, 1]$ will be located in the SEU-efficient region of the frontier.

In the presence of ambiguity, the optimal risk-return allocation will not always be located in the efficient part of the frontier because high incentives to self-hedge (i.e., $cov(H^*, h^*) > \kappa^{M2*}_{H,h;\theta}$) create a trade-off between the variance and ambiguity of wealth. Thus, the individual would appear to be "SEU-inefficient", however such action is optimal in the presence of the ambiguity dimension.

Proposition 5. Suppose that an ambiguity and risk averse agent is exposed to an ambiguous loss h and owns a risky non-traded asset H. Whenever the incentives to "*self-hedge*" are high (low) enough, i.e., $cov_{\bar{Q}}(H^*, h^*) > (<)\kappa^{M2*}_{H,h;\theta}$, the optimal mean-variance allocation $\{\sigma^2_{\bar{Q}}(W(\alpha^{M4*}_{amb})), E_{\bar{Q}}[W(\alpha^{M2*}_{amb})]\}$ induced by α^{M2*}_{amb} will be located on the SEU-inefficient (SEU-efficient) part of the frontier. More formally,

$$\frac{\mathrm{d}\,\sigma^2_{\bar{Q}}(W(\alpha))}{\mathrm{d}\,E_{\bar{Q}}[W(\alpha)]}\bigg|_{\alpha=\alpha^{M2*}_{amb}} = \begin{cases} > 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\theta} \\ = 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) = \kappa^{M2*}_{H,h;\theta} \\ < 0, & \text{if } cov_{\bar{Q}}(H^*, h^*) > \kappa^{M2*}_{H,h;\theta} \end{cases} \tag{19}$$

where $\kappa^{M2*}_{H,h;\theta} = \frac{mE_{\bar{Q}}[h]}{\gamma} \times \frac{\sigma^2_{\bar{Q}}(h^*)+\sigma^2_{\bar{Q}}(h^{\perp})}{\sigma^2_{\mu}(E[h^{\perp}])} > 0$.

Proof. See Appendix A.

To prove the proposition I show that the position of the SEU-efficient mean-variance allocation on the frontier depends on the incentives to "self-hedge" by comparing the optimal insurance demand with α^*_{MinVar}.

When the incentives to self-hedge are high enough ($cov(H^*, h^*) > \kappa^{M2*}_{H,h;\theta}$), the optimal risk-return allocation of an ambiguity averse individual will be located on the SEU-inefficient part of the frontier, because $\alpha^{M2*}_{amb} > \alpha^*_{MinVar}$. The individual would appear to be *SEU-inefficient* because she will buy more insurance than a SEU-maximizer would buy in the same situation. However, in the presence of ambiguity and aversion towards it, this "additional" demand for insurance is optimal.

Figure 2 shows an example of an optimal allocation that is seemingly SEU-inefficient but completely optimal when the ambiguity domain is taken into account. According to definition 2, the optimal insurance demand $\alpha_{amb}^{M2*} = 83.8\%$ (see tangency of isocurve and mean-variance frontier) would be SEU-inefficent because it is greater than $\alpha_{MinVar} = 70\%$. However, one has to take into account the ambiguity dimension.

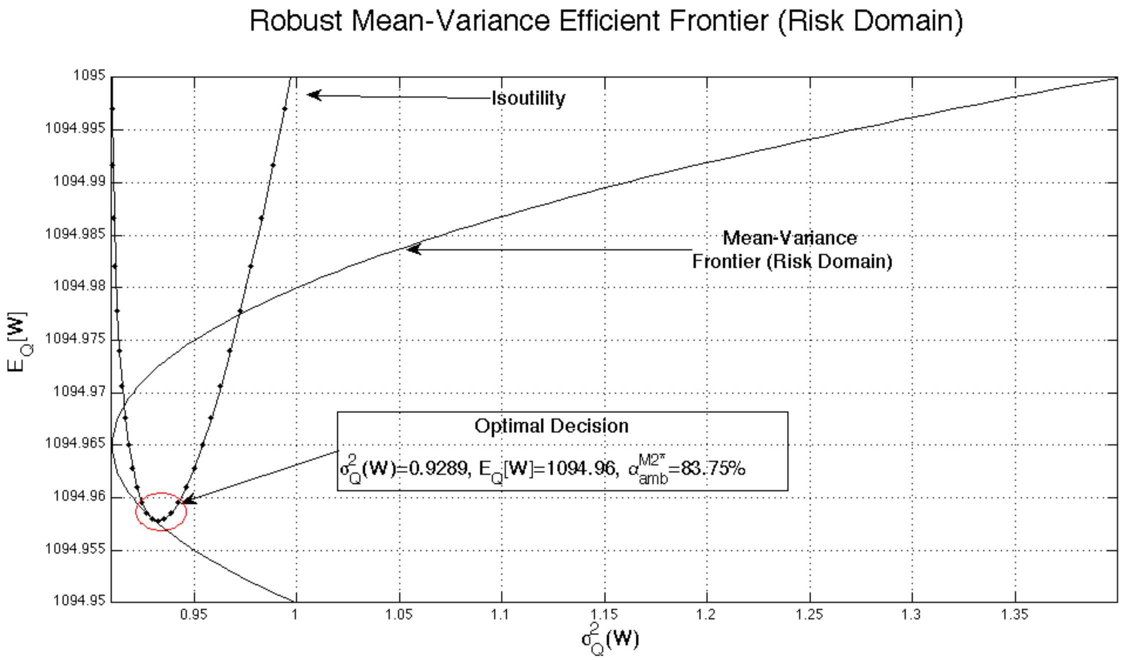

Figure 2: Optimal Choice in the Risky Domain in the Presence of Ambiguity. Parametric assumptions: $\theta = \gamma = 2, \sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp}) = \sigma^2_{\mu}(E[h^{\perp}]) = 1, W_0 = 1000, cov_{\bar{Q}}(H^*, h^*) = 0.3, m = .01, E_{\bar{Q}}[H] = 100$ and $E_{\bar{Q}}[h] = 5$.

Figure 3 shows the "mean-ambiguity" frontier that a subject faces in the presence of ambiguity as well as the isocurve that contains the optimal mean-ambiguity allocation $\{\sigma^2_{\mu}(W(\alpha_{amb}^{M2*}), E_{\bar{Q}}[W(\alpha_{amb}^{M2*})]\}$. If the individual was an SEU-maximizer, she would exhibit an insurance demand of $\alpha_{risk}^{M2*} = 67.5\%$. However, she would have to bear more ambiguity ($\sigma^2_{\mu}(W(\alpha_{risk}^{M2*})) = 0.11$) than she would be willing to accept if she was ambiguity averse ($\sigma^2_{\mu}(W(\alpha_{amb}^{M2*})) = 0.03$). Thus, when incentives to *self-hedge* are high ($cov_{\bar{Q}}(H^*, h^*) > \kappa_{H,h;\theta}^{M2*}$), the ambiguity averse individual will demand a level of insurance that seems excessive from an classical SEU perspective in order to reduced ambiguity to the desired level.

The exhibited SEU-inefficient type of behavior that I just described arises from a trade-off between risk and ambiguity that the DM must resolve. The intuition is that under

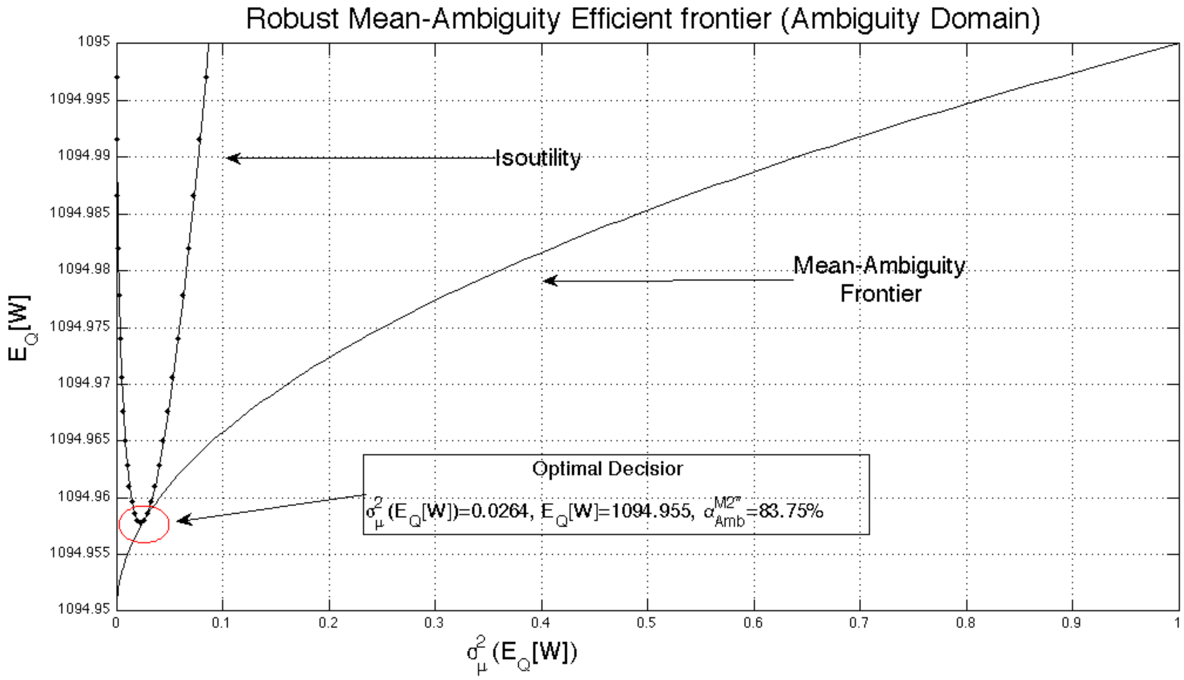

Figure 3: Optimal Choice in the Ambiguity Domain. Parametric assumptions: $\theta = \gamma = 2, \sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^\perp) = \sigma^2_\mu(E[h^\perp]) = 1, W_0 = 1000, cov_{\bar{Q}}(H^*, h^*) = 0.3, m = .01, E_{\bar{Q}}[H] = 100$ and $E_{\bar{Q}}[h] = 5$.

certain conditions, more insurance does not necessarily reduce variance and ambiguity at the same time. Figure 4 shows the locus of risk/ambiguity $\{\sigma^2_{\bar{Q}}(W), \sigma^2_\mu(E_Q[W])\}$ pairs that are attainable for any given level of insurance demand $\alpha \in [0, 1]$. An increase in the coinsurance demand moves the DM from right to left in the figure. For instance, Figure 4 emphasizes at the top right corner the case where the DM does not insure at all ($\alpha = 0$) and at the lower left corner the case when the DM is fully insured ($\alpha = 1$).

The parametrization for the example in the Figure 4 was carefully chosen to exemplify the situation in which it is not always possible to reduce variance and ambiguity at the same time. This is represented by the non-monotonicity of the curve in Figure 4. The first part of the curve that is decreasing indicates that more insurance decreases ambiguity but increases variance. The part of the figure that is increasing indicates that buying more insurance decreases both ambiguity and variance in that region. The non-monotonicity is present only when $cov_{\bar{Q}}(H^*, h^*) > 0$;[15] however, its mere presence does not imply that a DM will exhibit a

[15]It is easily shown that the curve defined by the locus of the $\{\sigma^2_{\bar{Q}}(W), \sigma^2_\mu(E_Q[W])\}$ pairs is non-monotonic if

SEU-inefficient type of behavior. We need the stronger condition that $cov_{\bar{Q}}(H^*, h^*) > \kappa^{M2*}_{H,h;\theta} > 0$, identified in proposition 5. Therefore, a DM will behave as if she was SEU-inefficient, but optimally if ambiguity is taken into account, when the incentives to self-hedge are sufficiently high.

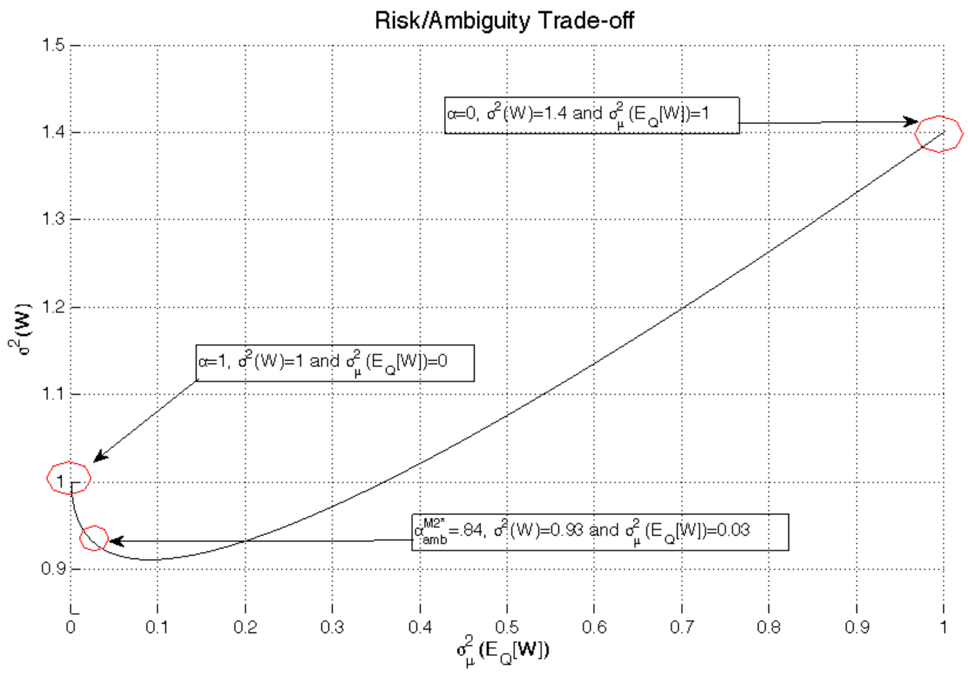

Figure 4: Risk/Ambiguity Trade-off and Optimal Choice. Parametric assumptions: $\theta = \gamma = 2, \sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp}) = \sigma^2_{\mu}(E[h^{\perp}]) = 1'W_0 = 1000, cov_{\bar{Q}}(H^*, h^*) = 0.3, m = .01, E_{\bar{Q}}[H] = 100$ and $E_{\bar{Q}}[h] = 5$. The figure shows the $\{\sigma^2_{\bar{Q}}(W), \sigma^2_{\mu}(E_Q[W])\}$ pairs that can be achieved for any value of coinsurance demand $\alpha \in [0, 1]$. Three cases are emphasized in the figure: (i) $\alpha = 1$, (ii) $\alpha = \alpha^{M2*}_{amb} = .8375$, and (iii) $\alpha = 0$.

What is the intuition for $\frac{\partial \alpha^{M2*}_{amb}}{\partial \theta} < 0$?

I can finally give a more comprehensive explanation for the intuitive result that more risk aversion might reduce the coinsurance demand in the presence of ambiguity and a non-traded asset. Proposition 3 and 5 show, respectively, that the same condition $(cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\theta})$ is needed for $\frac{\partial \alpha^{M2*}_{amb}}{\partial \theta}$ to be positive *and* the optimal mean-variance

$cov_{\bar{Q}}(H^*, h^*) > 0$. This is done by noting that $\frac{d\sigma^2_{\bar{Q}}(W)}{d\sigma^2_{\mu}(E_Q[W])} = \frac{1}{\sigma^2_{\mu}(E_Q[W])}\left(\sigma^2_{\bar{Q}}(W) - \frac{cov_{\bar{Q}}(H^*,h^*)}{(1-\alpha)}\right)$. The comparative static $\frac{d\sigma^2_{\bar{Q}}(W)}{d\sigma^2_{\mu}(E_Q[W])}$ is greater than 0 for any $\alpha \in [0, 1]$ if $cov_{\bar{Q}}(H^*, h^*) < 0$. If this covariance was positive, there is always a given coinsurance demand for which $\frac{d\sigma^2_{\bar{Q}}(W)}{d\sigma^2_{\mu}(E_Q[W])}$ is negative.

allocation to be on the SEU-efficient part of the frontier. The intuition is that in the SEU-efficient part of the frontier the lower variance required by higher risk aversion can be achieved by buying more insurance. This is possible because in this region of the frontier the variance of wealth is decreasing in the insurance demand α.[16]

Nevertheless, when the optimal risk-return allocation is in the SEU-inefficient part of the frontier, more risk aversion will decrease the demand for insurance. The reason is that the reduction in variance required by higher risk aversion cannot be met through more insurance under these conditions. When the individual buys more insurance, she reduces the variance of loss h she is exposed to, i.e., $(1-\alpha)^2\sigma^2_{\bar{Q}}(h)$. However, she also affects the *self-hedging* possibilities because more insurance reduces the effective covariance she actually faces, i.e., $(1-\alpha)cov(H^*, h^*)$.

When the optimal insurance demand is sufficiently high, such that $\alpha^{M2*}_{amb} > \alpha^*_{MinVar}$, a marginal increase in insurance also increases the variance of wealth because the reduction in self-hedging possibilities outweighs the reduction in the variance of h. Under these conditions, buying more insurance would increase the variance of wealth. Hence, an increment in risk aversion induces a reduction in the insurance demand. In the example shown in Figure 3, increasing the parameter of absolute risk aversion from 2 to 3 will decrease the optimal insurance demand from 83.8% to 81%.

Finally, one could assume that the subject perceives both the non-traded asset and the loss as ambiguous. Suppose the measure μ is a subjective joint distribution of beliefs about H and h and $cov_\mu(E[H^\perp], E[h^\perp])$ represents the incentives to *self-hedge* in the ambiguity domain. One can infer that it is possible to have similar versions of proposition 4 and 6 for the ambiguity domain. The bottom line is that that incentives to *self-hedge* are very important to insurance and asset demand both in the risk and ambiguity domain.

---

[16] $\frac{\partial \sigma^2_{\bar{Q}}(W)}{\partial \alpha} = -2(1-\alpha)\sigma^2_{\bar{Q}}(h) + 2cov(H^*, h^*) < 0$, if $\alpha < \alpha^*_{MinVar} = 1 - \frac{cov(H^*,h^*)}{\sigma^2_{\bar{Q}}(h)}$. By proposition 5, $\alpha^{M2*}_{amb} < \alpha_{MinVar}$ if $cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\theta}$. Thus, the variance of wealth is decreasing in the optimal insurance demand (i.e., $\frac{\partial \sigma^2_{\bar{Q}}(W)}{\partial \alpha}\big|_{\alpha=\alpha^{M2*}_{amb}} < 0$) whenever $cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\theta}$.

Trade-offs between Risk and Ambiguity in Portfolio Choice Problems

Cochrane [2007; p. 49] claims that in the multifactor efficient portfolio model developed by Fama [1996] "typical investors do not hold mean-variance efficient portfolios... [t]hey are willing to give up some mean-variance efficiency in return for a portfolio that hedges the state variable innovations." My model resembles the most simple case of such a model with one asset and one non-traded asset that plays the role of a factor. Proposition 5 suggests that there exist conditions under which ambiguity induces the subject to choose a SEU-inefficient mean-variance allocation that even takes into account the covariance of the factor with the other assets.

There is literature documenting the (SEU-)inefficiency in household portfolio and studying its source (e.g., Benartzi [2001]; Polkovnichenko [2005]; Goetzmann and Kumar [2008]). Typically, these studies propose a myriad of behavioral explanations for the under-diversification of portfolios like familiarity bias, ignorance, overconfidence, informational frictions, subjective beliefs and alternative preferences representations (e.g. Rank-Dependent utility theory).

However, there is limited research on the presence of non-traded assets as another possible explanation for the "seemingly" observed under-diversified portfolios, probably due to the difficulty of getting detailed data on households. Massa and Simonov [2006] study a unique data set of Swedish investors where they can identify various components of income, wealth and demographic characteristics. Massa and Simonov [2006; p. 661 and 667] find evidence that a wide range of investors do not hedge neither their financial assets nor their labor and entrepreneurial income (i.e., returns of non-traded assets). They conclude that familiarity, the tendency to invest in stocks that are geographically or professionally close to them, or that have been held for a long period, affects investors' hedging behavior and induces them to behave inefficiently from a classical SEU perspective. In particular, Massa and Simonov [2006] find that people tend to choose portfolio allocations that are positively correlated with their labor income. However, individuals with greater wealth tend to pick asset allocations that are more in the lines of traditional portfolio theory. Another explanation within the SEU model is that the subjective beliefs that investors use to make portfolio choices are different from the ones used by researchers to define their notion of efficiency. The explanation below also depends on subjective beliefs but departs from the SEU framework.

The issue of familiarity has a long tradition in experimental economics and was originally studied in contexts with ambiguity. Heath and Tversky [1991] and Fox and Tversky [1995] present experimental evidence in which they claim that subjects tend to prefer (or to avoid) ambiguous prospects for which they feel they are more (less) familiar. This familiarity interpretation was introduced as an explanation to typical behavior under ambiguity. However, many of the familiarity interpretations can be formalized in terms of attitudes towards ambiguity. In fact, studies such as Myung [2009] provide evidence to support the claim that ambiguity aversion is related to the equity market home bias paradox and Boyle et al. [2012] model the trade-offs between familiarity (modeled as attitudes towards ambiguity) and diversification.

A generalization of my model that includes traded assets, non-traded assets and insurance can offer an alternative explanation to the under-diversification puzzle of households. Intuitively, the interaction of traded assets, non-traded assets and insurable assets, that can each be ambiguous or not, can result in observed behavior that is "under-diversified" from a classical SEU perspective but optimal from a broader perspective. The main message is that deviations from the classical portfolio theory that assumes SEU can be explained by expanding the concept of portfolio to non-traded assets, which has been recognized in the financial literature, but more importantly to also allow for preference representations that can explain attitudes towards ambiguity.

I present below a simple model that can explain the following two observations in Massa and Simanov [2006]: (i) investors tend to hold stocks that are familiar to them and that are positively correlated with their labor income; and (ii) wealthy individuals have a greater tendency to pick stocks that can help hedging their labor income risk.

Suppose a DM (who could be an individual or a household) with labor income H, and that has the opportunity to invest part of his initial wealth W_0 into two financial assets, X and Y. For exposition purposes, assume that the DM perceives labor income H and asset X as risky, while asset Y is perceived as ambiguous. Labor income could be modeled as an ambiguous process but it is assumed risky for simplicity.[17] In the spirit of Massa and Simonov [2006], asset X can be understood as the set of stocks with which the DM is very familiar and asset Y is the set of stocks that are less familiar to the DM. There is a joint probability distribution $\bar{Q}$ that the DM considers is the best estimate to model the process driving the comovements of H, X and Y. The ambiguity of Y implies that the DM behaves as if there were more than one joint distributions, each denoted by Q_i. The interpretation of this particular case is that the individual behaves as if he was sure about the marginal distributions in the direction of H and X, but is unsure about the marginal distribution of Y. This results in the individual behaving as if there was a myriad of reasonable possible distribution Q_i with $i \in I$ and $\bar{Q}$ his best estimate of the joint distribution. The latter is calculated by taking the average over the many possible joint distributions with respect to a probability measure μ that captures the beliefs that any of the reasonable possible joint distributions is the true one.

The MMR orthogonal decomposition implies that labor income and assets can be expressed as $H = E_{\bar{Q}}[H] + H^*$, $X = E_{\bar{Q}}[X] + X^*$ and $Y = E_{\bar{Q}}[Y] + Y^* + Y^{\perp}$. The subindex $\bar{Q}$ is used in the expectations operator to remind the reader that the mean of the random variables is taken using the marginals of $\bar{Q}$. Define α and β as the proportions of initial wealth invested in asset X and Y, respectively, and there are no restrictions on these proportions. The final wealth can be written as $W = H + (1 - \alpha - \beta)W_0 + \alpha X + \beta Y$. Thus the maximization

[17]For example, take the case of a dentist for whom income varies from month to month but she is very familiar with the number of patients that might or might not come each month. In this case, the dentist might perceive her labor income as risky. In contrast, think of a self-employed consultant that is just starting up his business. He might have an idea of what a consultant can earn each month from the experience of others. However, this situation could be perceived as ambiguous because he doesn't have enough information to believe that his labor income is going to follow the path of other colleagues' income. In this case, the consultant might perceive his income as ambiguous.

problem is:

$$\max_{\alpha} E_{\bar{Q}}[W] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(W) - \frac{\gamma}{2}\sigma^2_{\mu}(E[W]) \tag{20}$$

where

$$E_{\bar{Q}}[W] = E_{\bar{Q}}[H] + (1-\alpha-\beta)W_0 + \alpha E_{\bar{Q}}[X] + \beta E_{\bar{Q}}[Y];$$

$$\sigma^2_{\bar{Q}}(W) = \sigma^2_{\bar{Q}}(H^*) + (\alpha)^2\sigma^2_{\bar{Q}}(X^*) + (\beta)^2[\sigma^2_{\bar{Q}}(Y^*) + \sigma^2_{\bar{Q}}(Y^{\perp})]$$

$$+2(\alpha\beta)cov_{\bar{Q}}(X^*, Y^*) + 2(\alpha)cov_{\bar{Q}}(X^*, H^*) + 2(\alpha)cov_{\bar{Q}}(Y^*, H^*)$$

$$\text{and } \sigma^2_{\mu}(E[W]) = (\beta)^2\sigma^2_{\mu}(E[Y^{\perp}])$$

Taking the derivative with respect to α and β, the first order conditions are:

$$[\alpha] : (E_{\bar{Q}}[X] - W_0) - \frac{\theta}{2}\left[2\alpha\sigma^2_{\bar{Q}}(X^*) + 2\beta cov_{\bar{Q}}(X^*, Y^*) + 2cov_{\bar{Q}}(X^*, H^*)\right] = 0 \tag{21}$$

$$\begin{aligned}[\beta] : (E_{\bar{Q}}[Y] - W_0) - \frac{\theta}{2}\left[2\beta\sigma^2_{\bar{Q}}(Y^*) + 2\alpha cov_{\bar{Q}}(X^*, Y^*) + 2cov_{\bar{Q}}(Y^*, H^*)\right] \\ - \frac{\gamma}{2}[2\beta\sigma^2_{\mu}(E[Y^{\perp}])] = 0\end{aligned} \tag{22}$$

For the sake of the argument assume that under probability $\bar{Q}$ asset X and Y only differ in that X is positively correlated with H:

$$cov_{\bar{Q}}(Y^*, H^*) = cov_{\bar{Q}}(X^*, Y^*) = 0,$$

$$cov_{\bar{Q}}(X^*, H^*) > 0,$$

$$E_{\bar{Q}}[X^*] = E_{\bar{Q}}[Y^*] = \bar{Z},$$

$$\sigma^2_{\bar{Q}}(X^*) = \sigma^2_{\bar{Q}}(Y^*) = \sigma^2.$$

Therefore, the optimal asset demands are given by:

$$\alpha^* = \frac{\bar{Z} - W_0}{\theta\sigma^2} - \frac{cov_{\bar{Q}}(X^*, H^*)}{\sigma^2} \tag{23}$$

$$\beta^* = \frac{\bar{Z} - W_0}{\theta\sigma^2 + \gamma\sigma^2_{\bar{Q}}(E[Y])} \tag{24}$$

An SEU maximizer should choose an asset allocation such that $\frac{\alpha^*}{\beta^*} < 1$ because both assets have the same mean and variance. However, asset Y is more suitable to hedge income risk than asset X is, given that $cov_{\bar{Q}}(Y^*, H^*) = 0$ and $cov_{\bar{Q}}(X^*, H^*) > 0$. This can be seen by taking the ratio of both asset demands and setting $\gamma\sigma^2_{\bar{Q}}(E[Y]) = 0$, since an SEU maximizer does not care about ambiguity.

However, an individual that is suffiently averse to ambiguity and/or finds asset Y very ambiguous, represented by a high value of $\gamma\sigma^2_{\bar{Q}}(E[Y])$, may choose a portfolio allocation that puts more weight on asset X (i.e., $\frac{\alpha^*}{\beta^*} > 1$) even though it is positively correlated with his income. This rationalizes the first empirical finding in Massa and Simonov [2006].

The second finding in Massa and Simonov [2006], that wealthy individuals behave according to what SEU would predict, can be explained in several ways. One common explanation is that wealthy investors are more financially sophisticated as suggested by Campbell [2006; p. 1576]. Another interpretation is that ambiguity attitudes are affected by the level of wealth. In my highly stylized model, an individual with sufficiently high wealth can overcome an aversion towards ambiguity and choose a portfolio allocation that puts more weight in the asset that helps to hedge income risk (i.e. $\frac{\alpha^*}{\beta^*} < 1$). This can be shown by taking the derivative of the ratio of asset demands with respect to initial wealth:

$$\begin{aligned}\frac{\partial \alpha^*/\beta^*}{\partial W_0} &= \left[\frac{\partial \gamma}{\partial W_0}\theta - \frac{\partial \theta}{\partial W_0}\gamma\right] \\ &- cov_{\bar{Q}}(X^*, H^*)\left[\left(\frac{\partial \theta}{\partial W_0} + \frac{\partial \gamma}{\partial W_0}\frac{\sigma^2_{\bar{Q}}(E[Y^\perp])}{\sigma^2}\right)(\bar{Z} - W_0) + \theta + \gamma\frac{\sigma^2_{\bar{Q}}(E[Y^\perp])}{\sigma^2}\right]\end{aligned} \tag{25}$$

For simplicity, assume that the DM exhibits CARA which implies that $\frac{\partial \theta}{\partial W_0} = 0$. Therefore, we can guarantee that $\frac{\partial \alpha^*/\beta^*}{\partial W_0} < 0$ by assuming the following: (i) $\frac{\partial \gamma}{\partial W_0} < 0$ which is implied by $\phi(.)$ satisfying the decreasing concavity property, that is to say, that absolute ambiguity aversion is decreasing in wealth; and (ii) that the estimated covariance between X and H is not so high, i.e. $cov_{\bar{Q}}(X^*, H^*) < \frac{Z - W_0}{\theta}$, otherwise strong hedging incentives would be the main explanation for the tilt of the portfolio towards the ambiguous asset. Therefore, an individual that originally chose a portfolio that was tilted towards the risky asset X and positively correlated to income can revert his choice towards a portfolio that puts more weight on the ambiguous asset Y if his wealth increases such that his aversion to ambiguity is significantly reduced.

3. Optimal Insurance Contracts Under Ambiguity

A. Optimality under Ambiguity of Insurance Contracts with a Straight Deductible: A Counterexample

A well-established result in insurance economics states that the optimal insurance contract for a risk averse DM is one with a straight deductible. The latter is due to Arrow [1971, p. 212], who proves the following proposition:

> If an insurance company is willing to offer any insurance policy against loss desired by the buyer at a premium which depends only on the policy's actuarial value, then the policy chosen by a risk-averting buyer will take the form of 100 per cent coverage above a deductible minimum.

Alary, Gollier and Treich [2010; p. 12-13] proves that this result is robust to the introduction of ambiguity when insurance decisions are made in isolation. This section shows that under certain conditions this result might not hold. I use a similar argument to Doherty [1984; p. 214], who showed that when a "portfolio contains non-insurable risky assets [e.g., human capital], the preference of the insured between a deductible and a coinsurance policy depends upon the covariance" between the loss and the non-tradable asset.

Proposition 6: In the presence of ambiguity and a non-traded asset, an insurance contract with a straight deductible does not always dominate a coinsurance contract.

Proof. See Appendix A.

The rationale of this result is as follows. The level of the deductible could be so high that there is a substantial loss in self-hedging possibilities that outweighs the benefits of variance and ambiguity reduction. However, in the presence of ambiguity, the conditions for a coinsurance arrangement to dominate are more stringent than in the case with only risk. The reason is that the potential benefits of self-hedging of the coinsurance arrangement must compensate both for the reduction in variance and ambiguity created by the contract with a deductible.

Notice that self-hedging is restricted in the present model to the risk domain. However, one can easily construct a model in which the non-traded asset is also ambiguous. This would make the self-hedging possibilities important in the ambiguity domain. The latter would be captured by a covariance term under the second order subjective measure μ of the ambiguous non-traded asset and the ambiguous loss. As a consequence, one can think of numerous trade-offs between the risk domain and the ambiguity domain.

B. The Borch Rule Under Ambiguity

An important result in the economic literature, derived by Borch [1962; p. 428], is the optimality of a coinsurance contract when both parties are risk averse. The rule is stated by Arrow[1971; p. 216] as follows:

> If the insured and the insurer are both risk averters and there are no costs other than coverage of losses, then any nontrivial Pareto-optimal policy, $[\alpha(h)]$, as a function of the loss, $[h]$, must have the property, $0 < [d\alpha/dh] < 1$. [18]

I show that this result is robust to the introduction of ambiguity aversion. However, the level of coinsurance under ambiguity, relative to the case with only risk, will depend on the relative importance of risk and ambiguity for each party and the differences in beliefs. In other words, it will depend on the ratios of the risk aversion and ambiguity aversion parameters, as well as on the compound distribution and the second order probability measure μ. I use similar arguments of the derivation in Arrow [1971, p. 217-219] and extend its logic to the ambiguity domain.

Proposition 7: Let $u(.)$ and $v(.)$[19] be the von-Neumann-Morgenstern utility functions that capture, respectively, the risk attitudes of the insured and the insurer. Assume both are risk averse, i.e. $u''(.) < 0$ and $v''(.) < 0$. Moreover, define $\phi_u(.)$ and $\phi_v(.)$ as the functions that capture the ambiguity aversion of the insured and the insurer, respectively. Both agents have a KMM representation of preferences under ambiguity. Let W_0 and W_1 be the initial wealth of

[18] I change the original notation in Arrow's quote to match ours. In his book, the loss is denoted by X and the insurance policy by I.

[19] Not to be confused with $v = \phi \circ u$ as defined in section 3.1.2.

the insured and the insurer. Additionally, define $\alpha(\tilde{h})$ as the insurance payment net of the insurance premium when the ambiguous loss takes the value $\tilde{h}$. The end-of-period wealth of each agent is:

- Insurer: $\widetilde{W_v} = W_1 - \alpha(\tilde{h})$
- Insured: $\widetilde{W_u} = W_0 - h + \alpha(\tilde{h})$

Each party's valuation of any insurance schedule $\alpha(\tilde{h})$ has a KMM structure:

- Insurer: $V = \int_{\Delta_v} \phi_v \left(\int_S v(\widetilde{W_v}) dQ_v \right) d\mu_v$
- Insured: $U = \int_{\Delta_u} \phi_u \left(\int_S u(\widetilde{W_u}) dQ_u \right) d\mu_v$

where the subindices imply the obvious definition for each variable.

Furthermore, assume a finite world where there are n possible outcomes for h, and $\bar{Q} = \{q_1, q_2, ..., q_i, ..., q_n\}$, θ, γ and $\mu_u = \{\mu_u^1, \mu_u^2, ..., \mu_u^k, ..., \mu_u^K\}$ represent, respectively, the compound probability distribution, risk attitudes, ambiguity attitudes and second order beliefs of the insured. Similarly, for the insurer these variables are represented by $\bar{P} = \{p_1, p_2, ..., p_n\}$, $\tilde{\theta}$, $\tilde{\gamma}$ and $\mu_v = \{\mu_v^1, \mu_v^2, ..., \mu_v^k, ..., \mu_v^K\}$. Notice that this is a general model that allows for different beliefs ($\bar{Q} \neq \bar{P}$), risk and ambiguity attitudes ($\theta \neq \tilde{\theta}$ and $\gamma \neq \tilde{\gamma}$) and probability measures over sets $\Delta_u = \{Q_1, Q_2, ..., Q_k, ..., Q_K\}$ and $\Delta_v = \{P_1, P_2, ..., P_k, ..., P_K\}$ ($\mu_u \neq \mu_v$). Elements in Q_k and P_k are denoted by q_j^k and p_j^k, respectively.

The risk/ambiguity arrangement in each of the possible outcomes h_i of the ambiguous loss is defined by

$$\frac{d\alpha(h_i)}{dh_i} = \frac{a\left[\theta q_i(1-q_i) + \gamma \sigma_{\mu_u}^2(q_i)\right]}{a\left[\theta q_i(1-q_i) + \gamma \sigma_{\mu_u}^2(q_i)\right] + b\left[\tilde{\theta} p_i(1-p_i) + \tilde{\gamma} \sigma_{\mu_v}^2(p_i)\right]} \tag{26}$$

Notice that $\frac{d\alpha(h_i)}{dh_i} \in (0,1)$, since $\theta > 0$, $\tilde{\theta} > 0$, $\gamma > 0$, $\tilde{\gamma} > 0$, and $p_i, q_i \geq 0$. In other words, the optimal contract is a *coinsurance* schedule.

Proof. See Appendix A

This implies that the Borch rule is robust to the introduction of ambiguity and even of heterogeneity of beliefs. However, the rule corrects for differences in beliefs, ambiguity and attitudes towards it. Though not reported here, the introduction of a risky non-traded asset does not affect the coinsurance structure of the optimal program. The intuition is that such an extension of the model will not affect the concavity of the Pareto-optimal frontier of the set of feasible $\{U, V\}$, leaving the structure of the maximization program unaffected.

For comparison purposes, $\frac{d\tilde{\alpha}(h_i)}{dh_i} = \frac{a\theta}{a\theta + b\tilde{\theta}}$ is the optimal coinsurance schedule with only risk and homogeneous beliefs. With $\bar{Q} = \bar{P}$ and $\mu_v = \mu_u = \mu$. The equivalent coinsurance rule when ambiguity is introduce is equal to:

$$\frac{a\left[\theta(1-q_i) + \gamma\sigma_\mu^2(q_i)/q_i\right]}{a\left[\theta(1-q_i) + \gamma\sigma_\mu^2(q_i)/q_i\right] + b\left[\tilde{\theta}(1-q_i) + \tilde{\gamma}\sigma_\mu^2(q_i)/q_i\right]} \tag{27}$$

For all h_i, the coinsurance demand is higher (lower) under ambiguity if $\frac{\tilde{\theta}}{\tilde{\gamma}} > (<) \frac{\theta}{\gamma}$. Intuitively, if the insurer is more ambiguity tolerant, relative to its own risk tolerance, than the insured is, the coinsurance demand is higher under ambiguity. However, this sharp comparative static is not possible when there are heterogeneous beliefs.

For any h_i, $\bar{Q} \neq \bar{P}$ and $\mu_v \neq \mu_u$, the optimal coinsurance demand is higher (lower) in the presence of ambiguity if $\frac{\tilde{\theta}}{\tilde{\gamma}} \frac{q_i(1-q_i)}{\sigma_{\mu_u}^2(q_i)} > (<) \frac{\theta}{\gamma} \frac{q_i(1-q_i)}{\sigma_{\mu_u}^2(q_i)}$. Consequently, when there are differences in beliefs and ambiguity matters to both parties, it cannot be easily predicted under which conditions the coinsurance demand is higher or lower than that under risk.

4. Conclusions

I have shown that if ambiguity matters, risk and ambiguity attitudes interact in nontrivial ways to determine the change of insurance demand for a given change in wealth. I derive sufficient conditions to guarantee that the optimal coinsurance demand is decreasing in wealth. Moreover, in the presence of a risky non-traded asset, I identify conditions under which more risk or ambiguity aversion decrease the demand for coinsurance.

Additionally, my model predicts behavior that is inconsistent with the classical portfolio theory that assumes Subjective Expected Utility theory, however, it provides hints to a possible solution of the under-diversification puzzle of households. The main message is that deviations from the traditional portfolio theory can be explained by expanding the concept of portfolio to non-traded assets, which has been recognized in the financial literature, but also by simultaneously allowing for preference representations that can explain attitudes towards ambiguity.

A modified Borch rule remains the optimal contract with bilateral risk/ambiguity aversion and heterogeneity in beliefs. However, an insurance contract with straight deductible might be dominated by a coinsurance schedule in the presence of ambiguity and a non-traded asset.

Several challenging questions remain. Other non-EU models of decision under ambiguity should be explored to corroborate that my results are not a mere artifact of the preference representation that I chose. However, an important challenge in the ambiguity literature is to derive more general results that do not depend heavily on the structure of the preference representation. Also, the optimality of contracts should be studied in the presence of asymmetric information and ambiguity. Finally, it would be desirable to derive a model which does not depend on an approximation and that allows for traded assets, non-traded assets and insurance.

References

Alary, David; Gollier, Christian and Treich, Nicolas, "The Effect of Ambiguity Aversion on Insurance Demand," *Working Paper*, Toulouse School of Economics, 2010.

Arrow, Kenneth J., "Liquidity Preference," LectureVI in *Lecture Notes for Economics 285, The Economics of Uncertainty*, Stanford University, 1963, 33-53.

Arrow, Kenneth J., *Essays in the Theory of Risk Bearing* (Chicago: Markham Publishing Company, 1971).

Athey, Susan C., "Monotone Comparative Statics Under Uncertainty," *Quarterly Journal of Economics*, 117(1), 2002, 187-223.

Baillon, Aurélien; L'Haridon, Oliver and Placido, Laetitia, "Ambiguity Models and the Machina Paradoxes',' *American Economic Review*, 101(4), 2010, 1547-1560.

Benartzi, Shlomo, "Excessive Extrapolation and the Allocation of 401(k) Accounts to Company Stock," *Journal of Finance*, 56(5), October 2001, 1747-1764.

Borch, Karl, "Equilibrium in a Reinsurance Market," *Econometrica*, 30(3), July 1962, 424-444.

Boyle, Phelim; Garlappi, Lorenzo; Uppal, Raman and Wang, Tan, "Keynes Meets Markowitz: The Tradeoff Between Familiarity and Diversification," *Management Science*, 58(2), February 2012, 253-272.

Camerer, Colin F. and Weber, Martin, "Recent Developments in Modeling Preferences: Uncertainty and Ambiguity," *Journal of Risk and Uncertainty*, 5, 1992, 325-370.

Cherbonnier, Frédéric and Gollier, Chrisitian, "Decreasing Aversion under Ambiguity," *Working Paper*, Toulouse School of Economics, 2011.

Cochrane, John H., "Portfolio Theory," *Working Paper*, University of Chicago, Graduate School of Business, 2007.

Collard, Fabrice; Mukerji, Sujoy; Sheppard, Kevin K. and Tallon, Jean-Marc, "Ambiguity and Historical Equity Premium," *Working Paper No. 550*, Department of Economics Discussion Paper Series, Oxford University, 2012.

de Finetti, Bruno, "Sulla Preferibilita," *Giornale degli Economisti e Annali di Economia*, 11, 1952, 685-709.

Doherty, Neil A. and Schlesinger, Harris, "The Optimal Deductible for an Insurance Policy When Initial Wealth is Random". *Journal of Business*, 56(4), 1983, 555-565.

Doherty, Neil A.(1984) "Portfolio Efficient Insurance Buying Strategies," *The Journal of Risk and Insurance*, 51(2), June 1984, pp. 205-224.

Eeckhoudt, Louis and Gollier, Christian, "Demand for Risky Assets and the Monotone Probability Ratio Order," *Journal of Risk and Uncertainty*, 11,1995, 113-122.

Eeckhoudt, Louis and Gollier, Christian, "The Effects of Changes in Risk on Risk Taking: A Survey," in G. Dionne (ed.), *Handbook of Insurance*, 2000, 131-151.

Ellsberg, Daniel, "Risk, Ambiguity and the Savage Axioms," *The Quarterly Journal of Economics*, 75(4), November 1961, 643-669.

Epstein, Larry G., "Three Paradoxes for the 'Smooth Ambiguity' Model of Preference," *Working Paper*, Boston University, 2010.

Fama, Eugene F., "Multifactor Portfolio Efficiency and Multifactor Asset Pricing," *Journal of Financial and Quantitative Analysis*, 31(4), December 1996, 441-465

Fishburn, Peter and Porter, R. Burr "Optimal Portfolios with One Safe and One Risky Asset: Effects of Changes in the Rate of Return and Risk," *Management Science*, 22, 1976, 1064-1073.

Fox, Craig R. and Tversky, Amos, "Ambiguity Aversion and Comparative Ignorance," *Quarterly Journal of Economics*, 110(3), August 1995, 585-603.

Gierlinger, Johannes and Gollier, Christian, Socially Efficient Discounting under Ambiguity Aversion. *Working Paper*, Toulouse School of Economics, 2010.

Goetzmann, William N. and Kumar, Alok, "Equity Portfolio Diversification," *Review of Finance*, 12(3), 2008, 433-463.

Gollier, Christian, "The Comparative Statics of Changes in Risk Revisited," *Journal of Economic Theory*, 66, 1995, 522-535.

Gollier, Christian, *The Economics of Risk and Time* (Cambridge, MA: The MIT Press, 2001).

Gollier, Christian, "Portfolio Choices and Asset Prices: The Comparative Statics of Ambiguity Aversion," *Working Paper*, Toulouse School of Economics, 2009.

Halevy, Yoram, "Ellsberg Revisited: An Experimental Study," *Econometrica*, 75(2), 2007, 503-536.

Heath, Chip and Tversky, Amos, "Preference and Belief: Ambiguity and Competence in Choice under Uncertainty," *Journal of Risk and Uncertainty*, 4, 1991, 5-28.

Ju, Nengjiu and Miao, Jianjun, "Ambiguity, Learning and Asset returns," *Econometrica*, forthcoming, 2012.

Klibanoff, Peter; Marinacci, Massimo and Mukerji, Sujoy, "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, 73(6), November 2005, 1849-1892.

L'Haridon, Oliver and Placido, Laetitia, "Betting on Machina's reflection example: An experiment on ambiguity," *Theory and Decision*, 69(3), 2009, 375-393.

Lintner, John, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, 47, February 1965, 13-37.

Maccheroni, Fabio; Marinacci, Massimo and Ruffino, Doriana, "Alpha as Ambiguity: Robust Mean-Variance Portfolio Analysis," *Working Paper*, Università Bocconi, 2011a.

Maccheroni, Fabio; Marinacci, Massimo and Ruffino, Doriana, "Does Uncertainty Vanish in the Small? The Smooth Ambiguity Case," *Working Paper*, Università Bocconi, 2011b.

Machina, Mark, "Risk, Ambiguity and the Rank-Dependent Axioms," *American Economic Review*, 99(1), 2009, 385-392.

Massa, Massimo and Simonov, Andrei "Hedging, Familiarity and Portfolio Choice," *Review of Financial Studies*, 19(2), Summer 2006, 633-685.

Mayers, David and Smith, Clifford W, "The Interdependence of Individual Portfolio Decisions and the Demand for Insurance," *Journal of Political Economy*, 91(2), April 1983, 304-311.

Mossin, Jan, "Equilibrium in a Capital Asset Market," *Econometrica*, 34, October 1966, 768-83.

Mossin, Jan, "Aspects of Rational Insurance Purchasing," *Journal of Political Economy*, Vol. 79, 1968, 553-568.

Myung, Noah, "Ambiguity Aversion in Asset Market: Experimental Study of Home Bias," *Working Paper No. 1306*, California institute of Technology, Division of Humanities and Social Sciences, 2009.

Pratt, John "Risk Aversion in the Small and in the Large," *Econometrica*, 32(1/2), January-April 1964, 122-136.

Rothschild, Michael and Stiglitz, Joseph, "Increasing Risk II: Its Economic Consequences," *Journal of Economic Theory*, 3, 1971, 225-243.

Savage, Leonard J., "*The Foundations of Statistics*" (New York: Dover Publications, 1972; Second Edition)

Schlesinger, Harris, "The Theory of Insurance Demand," in G. Dionne (ed.), *Handbook of Insurance*, 2000, 131-151.

Sharp, William F., "Capital Asset Prices: A Theory of Market Equilibrium under Conditions under Risk," *Journal of Finance*, 19, December 1964, 425-442.

Smith, Vernon, "Optimal Insurance Coverage," *Journal of Political Economy*, 76(1), 1968, 68-77.

Snow, Arthur, "Ambiguity Aversion and the Propensities for Self-Insurance and Self-Protection," *Journal of Risk and Uncertainty*, Vol. 42, No. 1, February 2011, 27-43.

Treich, Nicolas, "The Value of a Statistical Life under Ambiguity Aversion," *Journal of Environmental Economics and Management*, 59(1), January 2010, 15-26.

Vajda, Stefen, "Minimum Variance Reinsurance," *Astin Bulletin*, 1962.

Appendix A: Proofs of Results

Proof of Lemma 1. $\sigma^2_{\bar{Q}}(h) > \sigma^2_\mu(E[h])$. I show below that $\sigma^2_\mu(E[h]) = \sigma^2_{\bar{Q}}(h) - E_\mu[\sigma^2(h)]$, and hence the inequality holds.

$$\begin{aligned}
\sigma^2_\mu(E[h]) &= \int_\Delta (E[h])^2 d\mu - \left(\int_\Delta E[h] d\mu\right)^2 \\
&= \int_\Delta \left(\int_S h dQ\right)^2 d\mu - \left(\int_\Delta \int_S h dQ d\mu\right)^2 \\
&- \int_\Delta \int_S h^2 dQ d\mu + \int_\Delta \int_S h^2 dQ d\mu \\
&= \int_\Delta \int_S h^2 dQ d\mu - \left(\int_\Delta \int_S h dQ d\mu\right)^2 \\
&- \int_\Delta \left[\int_S h^2 dQ - \int_\Delta \left(\int_S h dQ\right)^2\right] d\mu \\
&= \sigma^2_{\bar{Q}}(h) - E_\mu[\sigma^2(h)]
\end{aligned}$$

Rearranging the last equality we have that $\sigma^2_{\bar{Q}}(h) = E_\mu[\sigma^2(h)] + \sigma^2_\mu(E[h])$. Since $E_\mu[\sigma^2(h)] > 0$ because it is an average of variances, the latter equation implies that $\sigma^2_{\bar{Q}}(h) > \sigma^2_\mu(E[h])$.

■

Proof of Proposition 1. Assume $m > 0$. The following derivative defines the comparative statics of the optimal insurance demand with respect to changes in initial wealth:

$$\frac{\partial \alpha^{M1*}_{amb}}{\partial W_0} = \frac{m E_{\bar{Q}}[h]}{\{\theta \sigma^2_{\bar{Q}}(h) + \gamma \sigma^2_\mu(E[h])\}^2} \sigma^2_{\bar{Q}}(h) \left[\frac{\partial \theta}{\partial W_0} + \frac{\partial \gamma}{\partial W_0} \frac{\sigma^2_\mu(E[h])}{\sigma^2_{\bar{Q}}(h)}\right] \tag{28}$$

where $\sigma^2_{\bar{Q}}(h) = \sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^\perp)$. Since $\gamma = \left\{-\frac{v''(W_0)}{v'(W_0)}\right\} - \left\{-\frac{u''(W_0)}{u'(W_0)}\right\}$ the partial derivative above can be written as:

$$\frac{\partial \alpha^{M1*}_{amb}}{\partial W_0} = \frac{m E_{\bar{Q}}[h]}{\{\theta \sigma^2_{\bar{Q}}(h) + \gamma \sigma^2_\mu(E[h])\}^2} \sigma^2_{\bar{Q}}(h) \left[\left(1 - \frac{\sigma^2_\mu(E[h])}{\sigma^2_{\bar{Q}}(h)}\right) \frac{\partial \theta}{\partial W_0} + \frac{\sigma^2_\mu(E[h])}{\sigma^2_{\bar{Q}}(h)} \frac{\partial \theta_v}{\partial W_0}\right] \tag{29}$$

where $\theta_v = -\frac{v''(W_0)}{v'(W_0)}$.

Notice that the sign of this partial derivative depends on the sign of the term in the square brackets. Since both u and v satisfy DC, which implies that θ_v and θ are decreasing in W_0, and $\sigma^2_{\bar{Q}}(h) > \sigma^2_{\mu}(E[h]) > 0$, the optimal insurance demand under ambiguity is decreasing in W_0, *i.e.* $\frac{\partial \alpha^{M1*}_{amb}}{\partial W_0} < 0$.

■

Proof of Lemma 2. The efficient (inefficient) part of the frontier is defined as the set of $\{\sigma^2_{\bar{Q}}(W(\alpha)), E_Q[W(\alpha)]\}$ pairs where $\frac{\mathrm{d}\,\sigma^2_{\bar{Q}}(W(\alpha))}{\mathrm{d}\,E_{\bar{Q}}[W(\alpha)]} > (<)0$.

This relationship can be found by taking the total derivative of the variance and expected value of wealth and dividing one over the other to obtain the following:

$$\frac{\mathrm{d}\,\sigma^2_{\bar{Q}}(W(\alpha))}{\mathrm{d}\,E_{\bar{Q}}[W(\alpha)]} = 2\frac{(1-\alpha)\sigma^2_{\bar{Q}}(h) - cov(H^*,h^*)}{mE_{\bar{Q}}[h]}$$

The sign of this derivative depends on the level of insurance demand:

$$\frac{\mathrm{d}\,\sigma^2_{\bar{Q}}(W(\alpha))}{\mathrm{d}\,E_{\bar{Q}}[W(\alpha)]} = \begin{cases} > 0, & \text{if } \alpha < \alpha^*_{MinVar} \\ = 0, & \text{if } \alpha = \alpha^*_{MinVar} \\ < 0, & \text{if } \alpha > \alpha^*_{MinVar} \end{cases}$$

■

Proof of Proposition 5. According to definition 2, the optimal allocation $\{\sigma^2_{\bar{Q}}(W(\alpha^{M2*}_{amb})), E_{\bar{Q}}[W(\alpha^{M2*}_{amb})]\}$ will be located on the SEU-inefficient part of the frontier whenever

$$\alpha^{M2*}_{amb} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta\sigma^2_{\bar{Q}}(h) + \gamma\sigma^2_{\mu}(E[h^{\perp}])} - \frac{cov_{\bar{Q}}(H^*,h^*)}{\sigma^2_{\bar{Q}}(h) + \frac{\gamma}{\theta}\sigma^2_{\mu}(E[h^{\perp}])} > 1 - \frac{cov_{\bar{Q}}(H^*,h^*)}{\sigma^2_{\bar{Q}}(h)} = \alpha^*_{MinVar}$$

Thus,

$$\alpha^{M2*}_{amb} > \alpha^*_{MinVar} \text{ if } cov_{\bar{Q}}(H^*,h^*) > \frac{mE_{\bar{Q}}[h]}{\gamma} \times \frac{\sigma^2_{\bar{Q}}(h^*) + \sigma^2_{\bar{Q}}(h^{\perp})}{\sigma^2_{\mu}(E[h^{\perp}])} = \kappa^{M2*}_{H,h;\theta}$$

Similarly, $\{\sigma^2_{\bar{Q}}(W(\alpha^{M2*}_{amb})), E_{\bar{Q}}[W(\alpha^{M2*}_{amb})]\}$ will be located on the SEU-efficient part of the frontier if $cov_{\bar{Q}}(H^*, h^*) < \kappa^{M2*}_{H,h;\theta}$.

■

Proof of Proposition 6. Suppose there is a DM that is both risk and ambiguity averse, owns a risky non-traded asset H and is exposed to an ambiguous loss h. Furthermore, suppose there is an insurance contract that covers 100% of any realization of the ambiguous loss above δ, a deductible amount chosen by DM. However, if $h < \delta$, the DM assumes the loss in its entirety. Moreover, assume that there is an alternative coinsurance contract that pays zero if no loss is realized and αh if a loss of amount h is realized, where $\alpha \in [0,1]$. The alternative coinsurance contract is carefully chosen such that a risk and ambiguity neutral insurer charges the same premium it does in the contract with deductible δ. Thus, α is defined by the following condition: $\alpha(1+m)E_{\bar{Q}}[h] = (1+m)E_{\bar{Q}}[max\{0, h-\delta\}] = \bar{\pi}$.

The strategy is to show that, under certain conditions, the CE implied by a coinsurance contract (CE_C, henceforth) can dominate the one implied by an actuarially equivalent contract with straight deductible (CE_D).

A coinsurance contract dominates one with a deductible if the following condition is satisfied:

$$\begin{aligned} CE_C \approx E_{\bar{Q}}[W_C] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(W_C) - \frac{\gamma}{2}\sigma^2_{\mu}(E[W_C]) > \\ E_{\bar{Q}}[W_D] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(W_D) - \frac{\gamma}{2}\sigma^2_{\mu}(E[W_D]) \approx CE_D \end{aligned} \tag{30}$$

where

$$W_C = H - h - \bar{\pi} + \alpha h,$$

$$W_D = H - h - \bar{\pi} + \max\{0, h-\delta\},$$

$$\sigma^2_{\bar{Q}}(W_C) = \sigma^2_{\bar{Q}}(H) + (1-\alpha)^2\sigma^2_{\bar{Q}}(h) - (1-\alpha)cov_{\bar{Q}}(H,h),$$

$$\sigma^2_{\bar{Q}}(W_D) = \sigma^2_{\bar{Q}}(H) + \sigma^2_{\bar{Q}}(\max\{h,\delta\}) - cov_{\bar{Q}}(H, \max\{h,\delta\})$$

W_C and W_D denote, respectively, the end-of-period wealth under the coinsurance contract and under the contract with straight deductible.

Given that the same premium $\bar{\pi}$ is charged, the expected value of wealth is the same under both contracts (i.e., $E_{\bar{Q}}[W_C] = E_{\bar{Q}}[W_D]$). Thus, we just need to focus on the variance-covariance structure and the perceived ambiguity generated by each contract.

Vajda [1962] showed that the deductible offers a greater reduction in the variance of retained losses given an amount of insurance premium. This implies that:

$$\frac{(1-\alpha)^2\sigma^2_{\bar{Q}}(h)}{[(1-\alpha)E_{\bar{Q}}[h]]^2} > \frac{\sigma^2_{\bar{Q}}(\max\{h,\delta\})}{[(E_{\bar{Q}}[\max\{h,\delta\}]]^2}$$

A similar argument can be made about the ambiguity perceived by the DM, since in my framework it is measured as the (subjective) variance of $E[h]$. Therefore, Vajda's argument also implies that an insurance contract with a straight deductible will induce lower perceived ambiguity than a coinsurance arrangement, given an expected value of retained losses. Therefore:

$$\frac{(1-\alpha)^2\sigma^2_{\mu}(E[h])}{[(1-\alpha)E_{\bar{Q}}[h]]^2} > \frac{\sigma^2_{\mu}(E[\max\{h,\delta\}])}{[E_{\bar{Q}}[\max\{h,\delta\}]]^2}$$

In the absence of the non-traded asset, these two simple facts show why Gollier's result holds. Intuitively, for a given level of insurance premium, such an insurance contract with straight deductible provides the DM with lower variance and ambiguity than the coinsurance arrangement does. However, as shown by Doherty [1984; 213-216] in a context with only risk, the covariance between the non-traded asset and the retained losses plays a crucial role in the robustness of Arrow's result. The same logic can be extended to the ambiguity domain.

A coinsurance arrangement dominates a contract with straight deductible if $CE_C > CE_D$. Using the MMR approximation of each CE, the latter condition is satisfied whenever

$$\frac{\gamma}{\theta}\left[(1-\alpha)^2\sigma_\mu^2(E[h]) - \sigma_\mu^2(E[\max\{h,\delta\}]\right] + \left[(1-\alpha)^2\sigma_{\bar{Q}}^2(h) - \sigma_{\bar{Q}}^2(\max\{h,\delta\})\right] \\ < (1-\alpha)cov(H,h) - cov(H,\max\{h,\delta\}) \tag{31}$$

It is plausible that, even though $cov(H,h)=0$, the covariance induced by the deductible, $cov(H,\max\{h,\delta\})$, is low enough such that the above condition is satisfied. In this case the coinsurance arrangement dominates the contract with a deductible.

■

Proof of Proposition 7. Let $u(.)$ and $v(.)$[20] be the von-Neumann-Morgenstern utility functions that capture, respectively, the risk attitudes of the insured and the insurer. Assume both are risk averse, i.e. $u''(.)<0$ and $v''(.)<0$. Moreover, define $\phi_u(.)$ and $\phi_v(.)$ as the functions that capture the ambiguity aversion of the insured and the insurer, respectively. Both agents have a KMM representation of preferences under ambiguity. Let W_0 and W_1 be the initial wealth of the insured and the insurer. Additionally, define $\alpha(\tilde{h})$ as the insurance payment net of the insurance premium when the ambiguous loss takes the value $\tilde{h}$. The end-of-period wealth of each agent is:

- Insurer: $\widetilde{W_v} = W_1 - \alpha(\tilde{h})$
- Insured: $\widetilde{W_u} = W_0 - h + \alpha(\tilde{h})$

Each party's valuation of any insurance schedule $\alpha(\tilde{h})$ has a KMM structure:

- Insurer: $V = \int_{\Delta_v} \phi_v\left(\int_S v(\widetilde{W_v})dQ_v\right)d\mu_v$
- Insured: $U = \int_{\Delta_u} \phi_u\left(\int_S u(\widetilde{W_u})dQ_u\right)d\mu_v$

where the subindices imply the obvious definition for each variable.

The set of all $\{U,V\}$ pairs has a boundary that is convex to the northeast. To show this, suppose any two insurance policies $\alpha_1(\tilde{h})$ and $\alpha_2(\tilde{h})$ that induce allocations $\{U_1,V_1\}$ and

[20]Not to be confused with $v=\phi\circ u$ as defined in section 3.1.2.

$\{U_2, V_2\}$, respectively. Define a third insurance schedule $\alpha_3(\tilde{h}) = .5\alpha_1(\tilde{h}) + .5\alpha_2(\tilde{h})$ for each $\tilde{h}$. The three insurance schedules induce the following end-of-period wealth W_{i1}, W_{i2} and $W_{i3} = .5W_{i1} + .5W_{i2}$ for $i = u, v$. Given that both parties are risk averse, the following statements are true:

$$u(W_{u3}) > .5u(W_{u1}) + .5u(W_{u2})$$

$$v(W_{v3}) > .5v(W_{v1}) + .5v(W_{v2})$$

Since both conditions hold for each $\tilde{h}$, it also holds when we take expectations. Without ambiguity, the latter argument will imply that the set of $\{U, V\}$ pairs induced by all the attainable insurance schedules will have a frontier that is convex to the northeast. Therefore, any Pareto-optimal insurance schedule can be obtained by maximizing a linear function $aE[u(\widetilde{W_u})] + bE[v(\widetilde{W_v})]$ for $a, b \geq 0$ and at least one positive. I show that this is also the case when ambiguity is introduced.

Suppose the sets of reasonably possible probabilities Δ_u and Δ_v are finite, and elements in each set are denoted by Q_{uk} and Q_{vk}, respectively, which are indexed by k. The second order belief μ_{uk} is the subjective probability weight that the insured puts over the probability distribution Q_{uk}; μ_{uk} is similarly defined for the insurer.The argument in the previous paragraph and the concavity of $\phi_u(.)$ and $\phi_v(.)$ imply that the following statements are true:

$\sum_k \mu_{uk} \times \phi_u(E_{Q_{uk}}[u(W_{u3})]) > .5 \times \sum_k \mu_{uk} \times \phi_u(E_{Q_{uk}}[u(W_{u1})]) + .5 \times \sum_k \mu_{uk} \times \phi_u(E_{Q_{uk}}[u(W_{u2})])$

or

$$U_3 > .5U_1 + .5U_2$$

and

$\sum_k \mu_{vk} \times \phi_v(E_{Q_{vk}}[v(W_{v3})]) > .5 \times \sum_k \mu_{vk} \times \phi_v(E_{Q_{vk}}[v(W_{v1})]) + .5 \times \sum_k \mu_{vk} \times \phi_v(E_{Q_{vk}}[v(W_{v2})])$

or

$$V_3 > .5V_1 + .5V_2$$

Since these statements hold for every pair of points $\{U_1, V_1\}$ and $\{U_2, V_2\}$ in the set of allocations defined by the possible insurance schedules, the northeast boundary of this set is convex to the northeast. Therefore, any Pareto-Optimal point, i.e. any point on the northeast boundary, can be obtained by maximizing a linear function $aU + bV$, for any $a, b \geq 0$, and at least one of them positive. Assume, that both agents have some bargaining power, thus $a, b > 0$. Moreover, I approximate the CE of U and V with the MMR approximation. Consequently, the maximization problem above is approximated by maximizing $a\widetilde{CE}_{insured} + b\widetilde{CE}_{insurer}$. Assuming that the state space S is finite, the optimal insurance schedule with bilateral risk and ambiguity aversion is the solution to:

$$\begin{aligned} \max_{\{\alpha(\tilde{h})\}} a &\left(E_{\bar{Q}}[W_0 - \tilde{h} + \alpha(\tilde{h})] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(\alpha(\tilde{h}) - \tilde{h}) - \frac{\gamma}{2}\sigma^2_{\mu_v}(E[\alpha(\tilde{h}) - \tilde{h}]) \right) \\ &+ b \left(E_{\bar{P}}[W_1 - \alpha(\tilde{h})] - \frac{\tilde{\theta}}{2}\sigma^2_{\bar{P}}(\alpha(\tilde{h})) - \frac{\tilde{\gamma}}{2}\sigma^2_{\mu_v}(E[\alpha(\tilde{h})]) \right) \end{aligned} \tag{32}$$

where $\bar{Q} = \{q_1, q_2, ..., q_i, ..., q_n\}$, θ, γ and $\mu_u = \{\mu_u^1, \mu_u^2, ..., \mu_u^k, ..., \mu_u^K\}$ represent, respectively, the compound probability distribution, risk attitudes, ambiguity attitudes and second order beliefs of the insured. Similarly, for the insurer these variables are represented by $\bar{P} = \{p_1, p_2, ..., p_n\}$, $\tilde{\theta}$, $\tilde{\gamma}$ and $\mu_v = \{\mu_v^1, \mu_v^2, ..., \mu_v^k, ..., \mu_v^K\}$. Notice that this is a general model that allows for different beliefs ($\bar{Q} \neq \bar{P}$), risk and ambiguity attitudes ($\theta \neq \tilde{\theta}$ and $\gamma \neq \tilde{\gamma}$) and probability measures over sets $\Delta_u = \{Q_1, Q_2, ..., Q_k, ..., Q_K\}$ and $\Delta_v = \{P_1, P_2, ..., P_k, ..., P_K\}$ ($\mu_u \neq \mu_v$). Elements in Q_k and P_k are denoted by q_j^k and p_j^k, respectively. Define the following:

$$E_{\bar{P}}[W_0 - \tilde{h} + \alpha(\tilde{h})] = W_1 - \textstyle\sum_j p_j \alpha(h_j),$$

$$E_{\bar{Q}}[W_0 - \tilde{h} + \alpha(\tilde{h})] = W_0 - \textstyle\sum_j q_j h_j + \sum_j q_j \alpha(h_j),$$

$$\sigma^2_{\bar{P}}(\alpha(\tilde{h})) = \textstyle\sum_j p_j \alpha(h_j)^2 - (\sum_j p_j \alpha(h_j))^2,$$

$$\sigma^2_{\bar{Q}}(\alpha(\tilde{h}) - \tilde{h}) = \textstyle\sum_j q_j (\alpha(h_j) - h_j)^2 - (\sum_j q_j (\alpha(h_j) - h_j))^2,$$

$$\sigma^2_{\mu_v}(E[\alpha(\tilde{h})]) = \textstyle\sum_k \mu_v^k [\sum_j p_j^k \alpha(h_j)]^2 - [\sum_k \mu_v^k \sum_j p_j^k \alpha(h_j)]^2$$

$$\sigma^2_{\mu_u}(E[\alpha(\tilde{h}) - \tilde{h}]) = \textstyle\sum_k \mu_v^k [\sum_j p_j^k (\alpha(h_j) - h_j)]^2 - [\sum_k \mu_v^k \sum_j p_j^k (\alpha(h_j) - h_j)]^2$$

The first order condition with respect to $\alpha(\tilde{h}_i)$ is:

$$b\left[-p_i - \tilde{\theta} p_i \left\{\alpha(h_i) - \sum_j p_j \alpha(h_j)\right\} - \tilde{\gamma}\left\{\sum_k \mu_v^k p_i^k (\sum_j p_j^k \alpha(h_j)) - p_i \sum_j p_j \alpha(h_j)\right\}\right]$$
$$+ a\left[-q_i - \theta q_i \left\{(\alpha(h_i) - h_i) - \sum_j q_j(\alpha(h_j) - h_j)\right\}\right] \tag{33}$$
$$+ a\left[-\gamma\left\{\sum_k \mu_u^k q_i^k (\sum_j q_j^k(\alpha(h_j) - h_j) - q_i \sum_j q_j(\alpha(h_j) - h_j)\right\}\right] = 0$$

Now differentiate the first order condition with respect to h_i:

$$-b\left[\tilde{\theta} p_i(1-p_i) + \tilde{\gamma}\sigma^2_{\mu_v}(p_i)\right]\left(\frac{d\alpha(h_i)}{dh_i}\right) - a\left[\theta q_i(1-q_i) + \gamma\sigma^2_{\mu_u}(q_i)\right]\left(\frac{d\alpha(h_i)}{dh_i} - 1\right) = 0$$

or equivalently

$$\frac{d\alpha(h_i)}{dh_i} = \frac{a\left[\theta q_i(1-q_i) + \gamma\sigma^2_{\mu_u}(q_i)\right]}{a\left[\theta q_i(1-q_i) + \gamma\sigma^2_{\mu_u}(q_i)\right] + b\left[\tilde{\theta} p_i(1-p_i) + \tilde{\gamma}\sigma^2_{\mu_v}(p_i)\right]} \tag{34}$$

Notice that $\frac{d\alpha(h_i)}{dh_i} \in (0,1)$, since $\theta > 0$, $\tilde{\theta} > 0$, $\gamma > 0$, $\tilde{\gamma} > 0$, and $p_i, q_i \geq 0$. In other words, the optimal contract is a *coinsurance* schedule.

■

Appendix B: Baseline Model: Insurance Demand under Risk

Consider a risk averse decision maker (DM) with a von Neumann-Morgenstern utility function $u(.)$, with $u'(.) > 0$ and $u''(.) < 0$, and initial wealth W_0. She is exposed to a risky loss h that has mean $E_P[h]$ and variance $\sigma_P^2(h)$ under probability distribution P. There is a risk neutral insurer that is willing to offer the individual any desired amount $\alpha \in (0, 1]$ of a coinsurance contract.[21] The insurer charges a premium $\pi = (1 + m)E_P[h]$ per unit of insurance, where $m \geq 0$ is the loading factor. The insurance contract pays zero if no loss is realized and αh if it is realized. If the individual decides to purchase insurance, her random final wealth is $W = W_0 - h - \alpha\pi + \alpha h$. Thus, the maximization problem of the individual is:

$$\max_\alpha E_P[u(W)] = \max_\alpha E_P[u(W_0 - h - \alpha(1 + m)E_P[h] + \alpha h)] \tag{35}$$

I simplify the problem by using the Arrow-Pratt approximation of the certainty equivalent (CE) of the expected utility wealth.[22] Therefore, the problem in equation (1) can be approximated by:

$$\max_\alpha E_P[W] - \frac{\theta}{2}\sigma_P^2(W) \tag{36}$$

subject to: $W = W_0 - h - \alpha\pi + \alpha h$

where

$$E_P[W] = E_P[W_0 - h - \alpha(1 + m)E_P[h] + \alpha h] = W_0 - (1 + \alpha m)E_P[h],$$

$$\sigma_P^2(W) = \sigma_P^2(W_0 - h - \alpha(1 + m)E_P[h] + \alpha h) = (1 - \alpha)^2\sigma_P^2(h)$$

and $\theta = -\frac{u''(W_0)}{u'(W_0)} > 0$ is the Arrow-Pratt degree of absolute risk aversion.

The first and second order condition with respect to α are:

$$[F.O.C] : -mE_P[h] - \frac{\theta}{2}[2(1 - \alpha)(-1)\sigma_P^2(h)] = 0 \tag{37}$$

[21] I focus in this section only on coinsurance contracts and deal later with the optimality of this type of contract.
[22] See Gollier [2001, pp. 21-24] for details of the approximation.

$$[S.O.C] : -\theta\sigma_P^2(h) < 0 \tag{38}$$

Solving for α, we obtain the following optimal insurance demand :

$$\alpha_{risk}^{M1*} = 1 - \frac{mE_P[h]}{\theta\sigma_P^2(h)} \tag{39}$$

Four standard results in the literature of insurance demand follow immediately:

1. Full coverage is optimal ($\alpha_{risk}^{M1*} = 1$), if coinsurance is available at a fair price ($m = 0$). This is the "Mossin-Smith Theorem" developed in Mossin [1968] and Smith [1968].

2. Given $m > 0$ such that $0 < \alpha_{risk}^{1*} < 1$, an increase in the degree of risk aversion will lead to an increase in the optimal demand for insurance at all levels of wealth, *ceteris paribus*. Schlesinger [2000; p. 138] proves this statement. In the present framework, this is easily seen from the following derivative:

$$\frac{\partial\alpha_{risk}^{M1*}}{\partial\theta} = \frac{mE_P[h]}{\theta^2\sigma_P^2(h)} > 0 \tag{40}$$

This result is a direct consequence of a standard result independently derived by Pratt [1964, p. 136) and Arrow [1971, p. 102]. They showed that an increase in absolute risk aversion decreases the demand for a risky asset. This would imply that a risk averse agent will demand more insurance to cover the loss than another individual that is less risk averse.[23]

A change in risk, $\sigma_P^2(h)$, affects coinsurance demand in the same direction that risk aversion does because:

$$\frac{\partial\alpha_{risk}^{M1*}}{\partial\sigma_P^2(h)} = \frac{mE_P[h]}{\theta[\sigma_P^2(h)]^2} > 0 \tag{41}$$

However, it has been shown that this is not a general result since it is likely that some

[23] In the context of insurance, the risky asset would be the retained loss. Thus, more insurance would translate into a lower exposure to the risky asset.

pessimistic deteriorations in beliefs increase the demand for a risky asset, thus decreasing insurance demand. Some important references that demonstrate this counterintuitive possibility are Rothschild and Stiglitz [1971], Fishburn and Porter [1976], Eeckhoudt and Gollier [1995], Gollier [1995] and Athey [2002].

3. Recall that if the local measure of risk aversion, $\theta = -\frac{u''(W_0)}{u'(W_0)}$, is decreasing (increasing) in W_0, individual's preferences are said to exhibit DARA (IARA). Moreover, if θ is independent of W_0, preferences exhibit CARA.

Proposition B.1(Schlesinger [2000; p.136]): Assume that $m > 0$ but is not too large such that $0 < \alpha_{risk}^{M1*} < 1$. Then, for an increase in the initial wealth W_0,

(i) α_{risk}^{M1*} will decrease under decreasing absolute risk aversion (DARA).

(ii) α_{risk}^{M1*} will be invariant under constant absolute risk aversion (CARA).

(iii) α_{risk}^{M1*} will be increasing under increasing absolute risk aversion (IARA).

Proof. Take the derivative of α_{risk}^{M1*} with respect to W_0:

$$\frac{\partial \alpha_{risk}^{M1*}}{\partial W_0} = \frac{mE_P[h]}{\theta^2 \sigma_P^2(h)} \frac{\partial \theta}{\partial W_0} \tag{42}$$

The sign of this derivative depends on whether preferences exhibit IARA, CARA or DARA:

$$\frac{\partial \alpha_{risk}^{M1*}}{\partial W_0} = \begin{cases} > 0, & \text{if } \frac{\partial \theta}{\partial W_0} > 0 \text{ (IARA)} \\ = 0, & \text{if } \frac{\partial \theta}{\partial W_0} = 0 \text{ (CARA)} \\ < 0, & \text{if } \frac{\partial \theta}{\partial W_0} < 0 \text{ (DARA)} \end{cases} \tag{43}$$

■

This proposition states that, if a subject exhibits DARA preferences, the optimal coinsurance demand decreases when initial wealth increases because her risk tolerance increases with it, that is to say, it becomes less risk averse. Similar intuitive arguments can be made about CARA and IARA. However, Schlesinger [2000; p. 136] warns that "each of these conditions [DARA, CARA and IARA] is shown to be sufficient for the comparative-static effects in Proposition [1], though not necessary."

4. **Definition B.1:** $\eta = -\frac{u'''(W_0)}{u''(W_0)} > 0$ is the index of absolute prudence. An individual is locally *prudent* at W_0 if $u'''(W_0) > 0$, which is equivalent to $u'(W_0)$ being locally convex (See Gollier [2001; p. 237]).

   Prudent individuals will increase savings if uncertainty affecting future income is introduced. In other words, the degree of prudence captures the sensitivity of savings to changes in risk. This is referred to as the *precautionary motive of saving*. The following proposition, derived by Gollier [2001; p. 238], links absolute risk aversion to the concept of prudence.

   Proposition B.2: Prudence is a *necessary* condition for preferences to exhibit DARA.

   Proof.

$$\begin{aligned}\frac{\partial\theta}{\partial W_0} &= -\left[\frac{u'''(W_0)u'(W_0) - [u''(W_0)]^2}{[u'(W_0)]^2}\right] \\ &= \left[-\frac{u''(W_0)}{u'(W_0)}\right]\left\{\left[-\frac{u''(W_0)}{u'(W_0)}\right] - \left[-\frac{u'''(W_0)}{u''(W_0)}\right]\right\} = \theta\left[\theta - \eta\right]\end{aligned} \tag{44}$$

   Given $\theta > 0$, $\eta > 0$ is a necessary condition for $\frac{\partial\theta}{\partial W_0} < 0$.

   ■

   Corollary B.1: Strong enough prudence (i.e., $\eta > \theta$) is a sufficient condition for $\frac{\partial\alpha_{risk}^{M1*}}{\partial W_0} < 0$. *Proof.* See Proposition 1 and 2.

Appendix C: Insurance Demand in the Presence of a Non-traded Asset under Risk

Suppose that the DM faces the same situation as in the baseline model, except that the individual now has a non-tradable and uninsurable asset (e.g., human capital) with risky return H that might be correlated with a risky loss h. Thus the risky end-of-period wealth is $W = W_0 + H - h - \alpha\pi + \alpha h$. Assume the DM maximizes expected utility defined by utility function $u(.)$ and $\bar{Q}$, the joint distribution of H and h. Therefore the maximization problem is:

$$\max_{\alpha} E_{\bar{Q}}[W] - \frac{\theta}{2}\sigma^2_{\bar{Q}}(W) \tag{45}$$

where

$$E_{\bar{Q}}[W] = E_{\bar{Q}}[W_0 + H - h - \alpha(1+m)E[h] + \alpha h] = W_0 + E_{\bar{Q}}[H] - (1+\alpha m)E_{\bar{Q}}[h]$$

$$\sigma^2_{\bar{Q}}(W) = \sigma^2_{\bar{Q}}(H) + (1-\alpha)^2\sigma^2_{\bar{Q}}(h) - 2(1-\alpha)cov_{\bar{Q}}(H,h)$$

The first order condition of the maximization problem is:

$$[\alpha] : -mE_{\bar{Q}}[h] - \frac{\theta}{2}\left\{2(1-\alpha)(-1)\sigma^2_{\bar{Q}}(h) - 2(-1)cov_{\bar{Q}}(H,h)\right\} = 0 \tag{46}$$

The optimal demand for insurance is:[24]

$$\alpha^{M3*}_{risk} = 1 - \frac{mE_{\bar{Q}}[h]}{\theta\sigma^2_{\bar{Q}}(h)} - \frac{cov_{\bar{Q}}(H,h)}{\sigma^2_{\bar{Q}}(h)} \tag{47}$$

The Mossin-Smith theorem only holds if $cov_{\bar{Q}}(H,h) \leq 0$. Doherty [1984; p. 209] derived a related result, and Mayers and Smith [1983] were the first to emphasize the interdependency between insurance demand and traded and non-traded assets.

Following the terminology in Mayers and Smith [1983, pp. 308], $cov_{\bar{Q}}(H,h)$ represents the individual's incentive to "*self-insure*." The sign of this covariance makes the insurance demand lower, equal or higher than in the absence of the non-traded asset (i.e., $\alpha^{M3*}_{risk} \lesseqqgtr \alpha^{M1*}_{risk}$).

[24]The second order condition is satisfied.

However, I prefer to use the term "*self-hedging*" to avoid confusion with the usage of "self-insurance" in the literature. The intuition is that even if the insurance premium is fair (i.e., $m = 0$), the DM might still not be willing to fully insure if she can compensate high losses with high realizations of her human capital. However, if $cov_{\bar{Q}}(H, h) < 0$, the individual might want to fully insure if a health shock (h) affects negatively her productivity, which would undermine her human capital.

Moreover, the introduction of a risky non-traded asset does not affect the Arrow-Pratt result. As a result, higher risk aversion increases the demand for insurance:

$$\frac{\partial \alpha_{risk}^{M3*}}{\partial \theta} = \frac{mE_{\bar{Q}}[h]}{\theta^2 \sigma_{\bar{Q}}^2(h)} > 0 \tag{48}$$

Finally, Proposition B.1 (the sufficiency of CARA, DARA or IARA for the insurance demand to be independent of, decreasing or increasing in W_0) and Proposition B.2 (the necessity of prudence to exhibit DARA) are robust to the presence of a non-traded asset. The optimal demands α_{risk}^{M1*} and α_{risk}^{M3*}, differ only in the term that depends on $\frac{cov_{\bar{Q}}(H,h)}{\sigma_{\bar{Q}}^2(h)}$, which is not affected by initial wealth. Thus, the proofs of these two propositions remain basically unchanged.