OPTIMAL ROBUSTNESS UNDER UNCERTAINTY

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(Very Preliminary)

ABSTRACT. Robustness concern has been long reflected in the decision models under uncertainty since the maxmin expected utility theory. All the models set the degree of robustness concern as fixed across all the payoff profiles. However, a decision maker's robustness concern may well changes when the certainty part or the unit scale of a payoff profile changes. This paper introduces formally a decision maker's robustness order, and characterize together a general class of robustness order and preference order over all the payoff profiles. The preference order has the feature of ranking the payoff profiles using the optimal degree of robustness. The optimal robustness level is endogenous and can change with the certainty part of a payoff profile and its unit scale.

1. INTRODUCTION

The seminal work of Ellsberg (1961) points out that there does not exist a prior on the state of the world to rationalize a decision maker(DM)'s preference over state-contingent payoff profiles. A major treatment in the literature is to represent the preference based on a set of priors rather than a single one. An important underlying feature along this line is its incorporation of a DM's robustness concern under uncertainty. By assigning a payoff profile the minimum expected utility over all the priors in a certain set, it takes into account the robustness of this evaluation against possible prior misidentification.

For example, the maxmin utility (MEU) (Gilboa and Schmeidler (1989)) has the form:

$$V(a) = \min_{p \in C} \int a dp$$

where a is a state-contigent payoff profile, p a probability measure of the states, and C a convex and compact subset of probability measures.

Constraint Preference (CP) (Hansen and Sargent (2001)) has the form:

$$V(a) = \min_{\{p: R(p||q) \leq \eta\}} \int adp$$

where q is a central prior, $R(p||q)$ the relative entropy of p with respect to q, and $\eta \in [0, \infty)$ a parameter. The relative entrophy $R(p||q)$ measures the "distance" between p and q, and $R(p||q) = 0$ if and only if $p = q$.

Multiplier preference (MP) (Hansen and Sargent (2001), Strzalecki (2011)) has the form:

$$V(a) = \min_{p \in \Delta} \{\int adp + \theta R(p||q)\}$$

where Δ is the set of all probability measures, and $\theta \in (0, \infty]$ a parameter.

Variational preference (VP) (Maccheroni, Marinacci and Rustichini (MMR) (2006)) has the form:

$$V(a) = \min_{p \in \Delta} \{\int adp + c(p)\}$$

where $c : \Delta \to [0, \infty]$ satisfies some conditions. When $c = \theta R(p||q)$, VP are reduced to MP. When $c(p) = 0$ if $p \in C$ and $c(p) = \infty$ if $p \notin C$, VP are red used to MEU.

However, a common feature of all these models is that the degree of robustness concern is fixed across all the payoff profiles. MEU fixes the set C; CP fixes η which determines how large the set of priors is under consideration. In MP and VP, the parameter θ and the function c are fixed. Both of them play a role of robustness measure. A big value of θ and a higher range of c corresponds to a small set of relevant priors, and thus a lower degree of robustness.

Fixed robustness level results in the scale invariance and translation invariance of the preferences. More precisely, in all these models, $I(a) \geq I(b)$ implies that $I(a+\bar{t}) \geq I(b+\bar{t})$ whenever $\bar{t}$ is a constant payoff profile producing t in each state. In MEU and CP, we have additionally that $I(a) \geq I(b)$ implies that $I(ta) \geq I(tb)$ whenever $t > 0$.

However, the robustness concern may well depend on the amount of the certainty part of a payoff profile and the unit scale. For example, consider the following two thought experiments.

TABLE 1.

$t \geq 0$	s_1	s_2
a_t	t+50	t+50
b_t	t	t+100

TABLE 2.

$t > 0$	s_1	s_2
a'_t	50t	50t
b'_t	0	100t

For a_t and b_t, one may prefer a_t to b_t when t is relatively small, and reverse the preference when t is very big. Indeed, given a large amount of ensured payoff, why not have a bet? For a'_t and b'_t, one may prefer b'_t to a'_t when t is small, and reverse the preference when t is very big. Because once s_1 happens, one can get nothing from b'_t while is guaranteed to get $30t$ from a'_t.

A similar example as the second one is also used in MMR (2006) to question MEU, while both VP and MEU are subject to the challenge of the first example.

This paper introduces formally the robustness concern under uncertainty as a binary relation $\succeq'$ over all the pairs (a, t) where a is a payoff profile and t a certainty utility level. More precisely, $(a, t) \succeq' (a', t')$ means that the DM feels more robust to achieve the utility level t in expectation from a than t' from a'. Under some axioms, the robustness depends on how large is the set of priors under which the payoff profile's expected utility achieves the certainty level. The certainty level is interpreted as the aspiration level under uncertainty. Thus, each payoff profile corresponds to a trade-off between the robustness and the aspiration level.

A DM may have a preference on the two criterion plane with robustness and aspiration measured in each dimension. He may evaluate a payoff profile by the optimal point on its robustness-aspiration trade-off line according to this preference. In this paper, we back up this preference from a DM's robustness order and his preference order over all the payoff profiles. We characterize together a general class of robustness orders, and preference orders which rank payoff profiles using the optimal degree of robustness. This optimal robustness degree is determined endogenously and can change with the amount of the certainty part of a payoff profile and its unit scale.

2. The Model

Let S be a finite set of states, and $\Delta(S)$ the set of all the probabilities on S. Let Z be a separable metric space denoting a set of outcomes, and $\Delta(Z)$ the set of all Borel probability measures on Z, equipped with the weak convergence topology. Note that $\Delta(Z)$ is also a separable metric space. An act f is a function from S into $\Delta(Z)$. Let $\mathcal{F} = \Delta(Z)^S$ be the set of all acts, equipped with the product topology. For any $p \in \Delta(Z)$, we identify it with the constant act in $\mathcal{F}$ sending p to each state. Preference $\succeq$ over acts is a binary relation on $\mathcal{F}$, with $\succ$ and $\approx$ denoting its asymmetric and symmetric parts respectively as usual.

The following axioms are considered for $\succeq$.

(I) (Order): $\succeq$ is complete and transitive.

(II) (Continuity): $\succeq$ is continuous.

(III) (Constant Certainty Independence): For any $p,\ q,\ r \in \Delta(Z)$ and any $\alpha \in [0,1]$, $p \approx q$ implies $\alpha p + (1-\alpha)r \approx \alpha q + (1-\alpha)r$.

(IV) (Monotonicity): For any $f,\ g \in \mathcal{F}$, $f(s) \succeq g(s)$ for all $s \in S$ implies $f \succeq g$.

(VI) (Nondegeneracy): $f \succ g$ for some $f,\ g \in \mathcal{F}$.

The following are some standard results in the literature.

Lemma 1. *Suppose that* $\succeq$ *satisfies (I) Order, (II) Continuity and (III) Constant Certainty Independence. Then there exists a continuous affine function* $u : \Delta(Z) \to \mathbb{R}$ *representing* $\succeq$ *restricted to* $\Delta(Z)$*, and* u *is unique up to positive affine transformation.*

Lemma 2. *Suppose that* $\succeq$ *satisfies (I) Order, (II) Continuity, (III) Constant Certainty Independence, (IV) Monotonicity and (VI) Nondegenaracy. Then there is a nonconstant continuous affine function* $u : \Delta(Z) \to \mathbb{R}$ *and a functional* $I : u(\Delta(Z))^S \to \mathbb{R}$ *(Note that* $u(\Delta(Z))$ *is an interval in* $\mathbb{R}$*.) such that*

(a) $f \succ g \Longleftrightarrow I(u(f)) \geq I(u(g))$;

(b) $I(p) = u(p)$ *for all* $p \in \Delta(Z)$*. Moreover,* u *is unique up to positive affine transformation.*

Now let us consider two main characters for decision making under uncertainty. On one hand, an uncertain act gives one the opportunity to win something so that it gives one the room for aspiring certain outcome. On the other hand, it does not guarantee one an ensured result, so it may be fragile in robustness to achieve his aspiration. For a given act, the more one aspire from it, the less robustness it can provide. There is a trade-off between aspiration and robustness under uncertainty.

To treat the aspiration and robustness concern fundamentally, we need to introduce another binary relation $\succeq'$ on $\mathcal{F} \times \Delta(Z)$, with $\succ'$ and $\approx'$ denoting its asymmetric and symmetric parts respectively. Given $f,\ g \in \mathcal{F}$ and $p,\ q \in \Delta(Z)$, $(f, p) \succeq' (g, q)$ means that the decision maker feels more robust to aspire something as good as p from f than something as good as q from g. To reflect the nature of this relation, let us consider the following axioms for $\succeq'$.

(i) (Order): $\succeq'$ is complete and transitive.

(ii) (Continuity): $\succeq'$ is continuous.

(iii) (Monotonicity): For any (f,p) $(g,q) \in \mathcal{F} \times \Delta(Z)$, if $m \in \Delta(S)$ and $\sum_{s\in S} f(s)m(s) \succeq p$ implies that $\sum_{s\in S} g(s)m(s) \succeq q$, then $(f,p) \preceq' (g,q)$. If in addition, there exists $m \in \Delta(S)$ such that $\sum_{s\in S} f(s)m(s) \prec p$ and $\sum_{s\in S} g(s)m(s) \succeq q$, then $(f,p) \prec' (g,q)$.

(iv) (Anonymity): For any $(f,p) \in \mathcal{F} \times \Delta(Z)$, if f' is a permutation of f, then $(f,p) \approx' (f',p)$.

Here are some useful facts.

Lemma 3. *Let $f,\ g \in \mathcal{F}$ and $p,\ q \in \Delta(Z)$ be given. Let $\max f(S) \in f(S)$ be such that $\max f(S) \succeq f(s)$ for all $s \in S$, and $\min f(S)$ is defined analogously.*

(a) Suppose that $\succeq$ satisfies (I) Order, (II) Continuity and (III) Constant Certainty Independence, and that $\succeq'$ satisfies (iii) Monotonicity. If $\max f(S) \prec p$, then $(f,p) \preceq (g,q)$. If $\min f(S) \succ p$, then $(f,p) \succeq (g,q)$. In particular, if $\max f(S) \prec p$ and $\max g(S) \prec q$, or $\min f(S) \succ p$ and $\min g(S) \succ q$, then $(f,p) \approx' (g,q)$.

(b) Suppose that $\succeq$ satisfies (I) Order, (II) Continuity and (III) Constant Certainty Independence, and that $\succeq'$ satisfies (iii) Monotonicity. If $p \succeq q$, then $(f,p) \preceq' (f,q)$. If $\max f(S) \succeq p \succ q \succeq \min f(S)$, then $(f,p) \prec' (f,q)$.

(c) Suppose that $\succeq'$ satisfies (i) Order and (ii) Continuity. Then there exists a continuous function r from $\mathcal{F} \times \Delta(Z)$ into $[0,1]$ representing $\succeq'$.

(d) Suppose that $\succeq$ satisfies (I) Order, (II) Continuity, (III) Constant Certainty Independence, (IV) Monotonicity and (VI) Nondegenaracy, and that $\succeq'$ satisfies (i) Order, (ii) Continuity and (iii) Monotonicity. Let $u : \Delta(Z) \to \mathbb{R}$ be given as in Lemma 2, and $r : \mathcal{F} \times \Delta(Z) \to [0,1]$ as in (c). Suppose that phi $= u(f)$. Define $r_\phi : u(\Delta(Z)) \to [0,1]$ such that $r_\phi(t) = r(f,p)$ if $u(p) = t$. Then r_ϕ is well defined. Moreover, we can choose r to be onto, so that $r_\phi(t) = 1$ when $t \leq \min \phi(S)$, and $r_\phi(t) = 0$ when $t \geq \max \phi(S)$.

Define $M(\phi,t) = \{m \in \Delta(S) | \sum_{s\in S}\phi(s)m(s) \geq t\}$ *for all* $(\phi,t) \in u(\Delta(Z))^S \times \mathbb{R}$. *Then* $r_\phi(t) \geq (>) r_{\phi'}(t')$ *whenever* $M(\phi,t) \supseteq (\supset) M(\phi',t')$. *In particular,* r_ϕ *is strictly decreasing on* $[\min\phi(S), \max\phi(S)]$.

Suppose further that $\succeq'$ *satisfies (iv) Anonymity, and that* ϕ' *is a permutation of* ϕ. *Then* $r_\phi = r_{\phi'}$.

Note that Anonymity makes sense when the depiction maker is facing total uncertainty and has no information about the probability over the states. This axiom "equalizes" two acts which come from a permutation of each other, and thus plays a role in accommodating Ellsberg-type behavior.

Next, let us consider two axioms on $\succeq$ which reflect the influence of $\succeq'$.

(VII) (Weak Dominace): For any $f,\ g \in \mathcal{F}$, if $(f,p) \succeq' (g,p)$ for all $p \in \Delta(Z)$, then $f \geq g$. If $(f,p) \prec' (g,p)$ for all p such that $\min f(S) \prec p \preceq \max f(S)$, then $f \prec g$.

(VIII) (Dominance): For any $K \in \mathbb{N}$ and any $f, g_1, ..., g_K \in \mathcal{F}$, if for any p such that $\min f(S) \prec p \preceq \max f(S)$, there exists $k \in \{1, ..., K\}$ such that $(f,p) \preceq' (\prec')(g,p)$, then $f \preceq (\prec) \max\{g_1, ..., g_K\}$.

Lemma 4. *(a) If* $\succeq'$ *satisfies (i) Order, (ii) Continuity and (iii) Monotonicity, then (VII) Dominance implies (VIII) Weak Dominance.*

(b) If $\succeq$ *satisfies (I) Order, (II) Continuity, (III) Constant Certainty Independence, and* $\succeq'$ *satisfies (iii) Monotonicity, then* $\succeq$ *satisfies (IV) Monotonicity.*

Here is the main result.

Theorem 1. *The following two statements are equivalent.*

(1) Preference relation $\succeq$ *satisfies (I) Order, (II) Continuity, (III) Constant Certainty Independence, (VI) Nondegenaracy and (VIII) Dominance. Robustness relation satisfies (i) Order, (ii) Continuity and (iii) Monotonicity.*

(2) There exist a nonconstant continuous affine function $u : \Delta(Z) \to \mathbb{R}$, *for each* $\phi \in u(\Delta(Z))^S$, *a continuous, decreasing and surjective function* $r_\phi : u(\Delta(Z)) \to [0,1]$ *which is strictly decreasing on*

$[\min\phi(S), \max\phi(S)]$, *and an upper semicontinuous function* V *on* $u(\Delta(Z)) \times [0,1]$ *such that for any* $f,\ g \in \mathcal{F}$ *and* $p,\ q \in \Delta(Z)$,

(a) $f \succeq g \iff \max_{t\in[\min u(f(S)),\max u(f(S))]} V(t, r_{u(f)}(t)) \geq \max_{t\in[\min u(g(S)),\max u(g(S))]} V(t, r_{u(g)}(t))$;

(b) $(f,p) \succeq' (g,q) \iff r_{u(f)}(u(p)) \geq r_{u(g)}(u(q))$.

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To be added...