Foundations for Prospect Theory Through Probability Midpoint Consistency

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Abstract: For the famous prospect theory model there is hitherto no preference foundation for general sets of outcomes. All existing models assume a rich structure for the set of outcomes and propose preference conditions that hinge upon that structure. Yet in many important applications where prospect theory is assumed, like health or insurance, the set of outcomes is degenerate. In these more general settings it is unclear what preference conditions are required, beyond the standard assumptions, to pin down prospect theory. This paper proposes a consistency principle for elicited probability midpoints that requires a consistent treatment of probabilities of gains and similarly a consistent treatment of probabilities of losses. We show that, in the presence of the other standard preference conditions, this consistency principle implies prospect theory.

Keywords: probability midpoint, prospect theory, rank-dependence, reference-dependence, sign-dependence.

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1 Introduction

Kahneman and Tversky (1979) provided us with a powerful descriptive theory for decision under risk. Later, in Tversky and Kahneman (1992), prospect theory[2] (PT) was extended to uncertainty and ambiguity by incorporating the requirements of rank-dependence introduced by Quiggin (1981, 1982) and Schmeidler (1989), and received a sound preference foundation by using the tools underlying continuous utility measurement developed in Wakker (1989) (see also Wakker and Tversky 1993). As PT can handle sign-dependent probabilistic risk attitudes, ambiguity attitudes, reference-dependence, loss aversion and diminishing sensitivity of utility, it became the most popular and well-known descriptive theory for risk and uncertainty (Starmer 2000, Kahneman and Tversky 2000, Wakker 2010). Yet, since its original form, it took many years before the first foundations of PT for decision under risk was made readily available (Chateauneuf and Wakker 1999). Recently, Kotyal, Spinu and Wakker (2011) provided foundations of PT for continuous distributions. All these theoretical developments assumed that the set of outcomes is endowed with a sufficiently rich structure that then allows for the derivation of continuous cardinal utility.

This paper takes a different approach. It does not assume richness of the set of outcomes, and instead it follows the traditional approach pioneered by von Neumann and Morgenstern (1944) of using the structure naturally given by the probability interval. This approach has been used to derive preference foundations for rank-dependent utility (RDU) by Chateauneuf (1999), Abdellaoui (2002) and more recently by Zank (2010); specific parametric probability weighting functions were provided by Diecidue, Schmidt and Zank (2009), Abdellaoui, l'Haridon and Zank (2010) and Webb and Zank (2011). Neither of these have looked at PT-preferences.[3] Indeed, these late discoveries and preference foundations for RDU may explain why hitherto PT has not been derived using this "probabilistic

[2]Some authors prefer to distinguish the original prospect theory of Kahneman and Tversky (1979) from the modern version, cumulative prospect theory, of Tversky and Kahneman (1992). Indeed, as Wakker (2010, Apendix 9.8) clarifies, the models make different predictions, in general. Here we restrict attention to the modern version, and hence, we use the shorter name *prospect theory*.

[3]An exception is Prelec (1998), where PT is assumed, however, the key preference condition there requires a continuous utility.

approach." Our aim is to fill this gap and to show that PT can be obtained from preference conditions that rely solely on there being given the objective probabilities for outcomes while the set of outcomes can be very general.

The importance of having sound preference foundations for decision models, and in particular for PT, has recently been reiterated by Kotyal, Spinu and Wakker (2011, pp. 196–197). If a continuous utility is not available, because the outcomes are discrete (e.g., as in health or insurance), the relationship between the empirical primitive (i.e., the preference relation) and the assumption of PT become blurred, which is undesirable. In that case one can no longer be sure that our predictions and estimates are in line with the behavior induced by the preference relation. Our preference conditions are necessary and sufficient for PT and, therefore, they help clarifying which assumptions one makes by invoking the model.

Our key preference condition is based on the idea of probability midpoint elicitations. It requires that elicited probability midpoints are independent of the outcomes (i.e., the stimuli) used to derive these midpoints, whenever all outcomes are of the same sign (i.e., either all outcomes are gains or all are losses). Indeed, under PT the probability weighting function for gain probabilities may be different to the probability weighting function for probabilities of losses, that is, we have *sign-dependence*. Many empirically studies provided evidence on sign-dependence (Edwards 1953, 1954, Hogarth and Einhorn 1990, Tversky and Kahneman 1992, Abdellaoui 2000, Bleichrodt, Pinto, and Wakker 2001, Etchard-Vincent 2004, Payne 2005, Abdellaoui, Vossmann and Weber 2005, Abdellaoui, l'Haridon and Zank 2010).[4] We therefore invoke consistency of probability midpoints for gains and consistency of probability midpoints for losses. It turns out that this principle of consistency is sufficient to obtain PT in the presence of some standard preference conditions.

The original elicitation technique for nonparametric probability weighting functions was presented

[4] Sign-dependence is one of the consequences of reference dependence. The latter serves as the key explanation for prominent phenomena like the disparity between willingness to pay and willingness to accept (Kahneman, Knetsch, and Thaler 1990, Bateman et al. 1997, Viscusi, Magat and Huber 1987, Viscusi and Huber 2012), the endowment effect (Thaler 1980, Loewenstein and Adler 1995), and the status quo bias (Samuelson and Zeckhauser, 1988).

in Abdellaoui (2000) and Bleichrodt and Pinto (2000). These papers invoke utility measurements prior to the elicitation of probability weighting functions and, hence, still assume and use continuous utility. A simplified version of the method appeared recently in van de Kuilen and Wakker (2011). The latter method, which also applies to the study of ambiguity attitude, that is, when uncertainty is not described by objective probabilities, requires only one single utility midpoint elicitation so needs continuous utility for that but no more. In contrast, the method of Wu, Wang and Abdellaoui (2005) can be applied in the probability triangle and does not necessitate utility midpoint elicitation. We assume probabilities are given and extend these methods to axiomatically derive PT. This way we obtain preference conditions that are empirically meaningful and in turn behavioral foundations for PT for decision under risk.

Next we present preliminary notation and recall the standard preference condition of preferences and some implications thereof. In Section 3 we elaborate and present our main preference condition and the main theorem. Extensions are discussed in Section 4 and the concluding remarks in Section 5 are followed by an Appendix with proofs.

2 Preliminaries

In this section we recall the standard ingredients for decision under risk and the traditional preference conditions that are shared by expected utility and prospect theory.

2.1 Notation

Let X denote the set of *outcomes*. For simplicity of exposition, we make several assumptions that will later, in Section 4, be relaxed to demonstrate the full generality of our approach. First, we assume a finite set of outcomes, such that $X = \{x_1, \ldots, x_n\}$, with $n > 4$. A *prospect* is a finite probability distribution over the set X. Prospects can be represented by $P = (\tilde{p}_1, x_1; \ldots; \tilde{p}_n, x_n)$ meaning that outcome $x_j \in X$ is obtained with probability $\tilde{p}_j$, for $j = 1, \ldots, n$. Naturally, $\tilde{p}_j \geq 0$ for $j = 1, \ldots, n$

and $\sum_{i=1}^{n} \tilde{p}_i = 1$. Let $\mathcal{L}$ denote the set of all prospects.

A *preference relation* $\succcurlyeq$ is assumed over $\mathcal{L}$, and its restriction to subsets of $\mathcal{L}$ (e.g., all degenerate prospects where one of the outcomes is received for sure) is also denoted by $\succcurlyeq$. The symbol $\succ$ denotes strict preference while $\sim$ denotes indifference ($\preccurlyeq$ and $\prec$ denote reversed weak and strict preferences, respectively).

Our second expositional simplification is the assumption that no two outcomes in X are indifferent, and further, that outcomes are ordered from best to worst, i.e., $x_1 \succ \cdots \succ x_n$. We assume that X contains a reference point: for some $k \in \{1, \ldots, n\}$ let x_k be this reference point such that all outcomes $x \succ x_k$ are *gains* and all outcomes $x_k \succ x$ are *losses*. We assume that X contains at least two gains and at least two losses.

It will be convenient to use an alternative notation for prospects, following Abdellaoui (2002) and Zank (2010). In the *(de)cumulative probabilities* notation $P = (p_1, \ldots, p_n)$, where $p_j = \sum_{i=1}^{j} \tilde{p}_i$ denotes the probability of obtaining outcome x_j or better, $j = 1, \ldots, n$.[5] Obviously, $p_n = 1$. Note that we have dropped outcomes from the cumulative probability notation for prospects to further simplify the exposition.

Recall, that under *expected utility* (EU) prospects are evaluated by

$$EU(p_1, \ldots, p_n) = \sum_{j=1}^{n} [p_j - p_{j-1}] u(x_j), \tag{1}$$

with a *utility* function, u, which assigns to each outcome a real number and is monotone (that is, u agrees with the preference ordering over outcomes: $u(x_i) \geq u(x_j) \Leftrightarrow x_i \succcurlyeq x_j$).[6] Under EU the utility is *cardinal*, i.e., it is unique up to multiplication by a positive constant and translation by a location parameter.

A more general model is *rank-dependent utility* (RDU) where a prospect $P = (p_1, \ldots, p_{n-1}, 1)$ is

[5] Similarly, in the *cumulative probabilities* notation $P = (1, 1 - p_1, \ldots, 1 - p_{n-1})$ where entries denote the probability of obtaining outcome x_j or less, $j = 1, \ldots, n$.

[6] Here and elsewhere we use the convention that summation over zero elements is equal to 0. E.g., $p_0 = \sum_{j=1}^{0} \tilde{p}_j = 0$.

evaluated by

$$RDU(p_1, \ldots, p_n) = \sum_{j=1}^{n} [w(p_j) - w(p_{j-1})]u(x_j). \tag{2}$$

Utility is similar to EU, however, RDU involves a weighting function for probabilities, w, that is uniquely determined. Formally, a *weighting function*, w, is a mapping from the probability interval $[0, 1]$ into $[0, 1]$ that is strictly increasing with $w(0) = 0$ and $w(1) = 1$. In this paper the axiomatically derived weighting functions are continuous on $[0, 1]$. There is, however, empirical and theoretical interest in discontinuous weighting functions at 0 and at 1 (Kahneman and Tversky 1979, Birnbaum and Stegner 1981, Bell 1985, Cohen 1992, Wakker 1994, 2001, Chateauneuf, Eichberger and Grant 2007, Webb and Zank 2011). We discuss relaxing continuity at the extreme probabilities in Section 4.

The model of interest in this paper extends RDU by incorporating reference-dependence. This has the implication that the weighting function will depend on the whether the cumulative probabilities are those of gains or whether they are probabilities of losses. For this reason the term sign-dependence is used to highlight that the nonlinear treatment of probabilities depends on the sign of the outcome attached to that probability. Under *Prospect Theory* (PT) a prospect $= (p_1, \ldots, p_n)$ is evaluated by

$$PT(p_1, \ldots, p_n) = \sum_{j=1}^{k-1} [w^+(p_j) - w^+(p_{j-1})]u(x_j) + \sum_{j=k+1}^{n} [w^-(1 - p_{j-1}) - w^-(1 - p_j)]u(x_j), \tag{3}$$

where $u(x_k) = 0$ is assumed and w^+ and w^- are the continuous and strictly increasing probability weighting functions for cumulative probabilities of gains and losses, respectively. Under PT the utility is a *ratio scale* (i.e., it is unique up to multiplication by a positive constant) and the weighting functions are uniquely determined.

The definition of the probability weighting functions over cumulated probabilities starting with the largest gain (loss) is the familiar presentation of the PT-formula for evaluating prospects. Sometimes, in the literature, the dual of the weighting function for probabilities of losses is employed: $\tilde{w}^-(p) = 1 - w^-(1 - p), p \in [0, 1]$. Then, the decision weight $[w^-(1 - p_{j-1}) - w^-(1 - p_j)]$ for a loss $x_j, j > k$, in Equation (3) can be rewritte as $[\tilde{w}^-(p_j) - \tilde{w}^-(p_{j-1})]$.

As mentioned in the introduction several preference foundations for PT have been proposed using the approach based on continuous utility. Foundations with general continuous utility include Tversky and Kahneman (1992), Wakker and Tversky (1993), Chateauneuf and Wakker (1999), Köbberling and Wakker (2003, 2004), Wakker (2010) and Kotyal, Spinu and Wakker (2011). Derivations of CPT with specific forms for the utility function (linear/exponential, power, and variants of multiattribute utility) have been provided in Zank (2001), Wakker and Zank (2002), Schmidt and Zank (2009). Bleichrodt, Schmidt and Zank (2009) assume attribute specific reference points for derivations of functionals that combine PT and multiattribute utility. In the next subsection we present the standard preference conditions that all evaluation functionals presented in this section have to satisfy.

2.2 Traditional Preference Conditions

This subsection presents the classical preference conditions that are necessary for RDU and PT. We are interested in conditions for a preference relation, $\succcurlyeq$, on the set of prospects $\mathcal{L}$ in order to *represent* the preference relation by a function V. That is, V assigns to each prospect a real value, such that for all $P, Q \in \mathcal{L}$,

$$P \succcurlyeq Q \Leftrightarrow V(P) \geq V(Q).$$

A requirement for such a representation V is that $\succcurlyeq$ is a *weak order*, i.e., the following axiom holds:

WEAK ORDER: The preference relation $\succcurlyeq$ is *complete* ($P \succcurlyeq Q$ or $P \preccurlyeq Q$ for all $P, Q \in \mathcal{L}$) and transitive.

Further requirements are those of first order stochastic dominance and of continuity in probabilities.

DOMINANCE: The preference relation satisfies *first order stochastic dominance* (or *monotonicity* in cumulative probabilities) if $P \succ Q$ whenever $p_j \geq q_j$ for all $j = 1, \ldots, n$ and $P \neq Q$.

Continuity: The preference relation $\succcurlyeq$ satisfies *Jensen-continuity* on the set of prospects $\mathcal{L}$ if for all prospects $P \succ Q$ and R there exist $\rho, \mu \in (0,1)$ such that $\rho P + (1-\rho)R \succ Q$ and $P \succ \mu R + (1-\mu)Q$.[7]

A monotonic weak order that satisfies Jensen-continuity on $\mathcal{L}$ also satisfies the stronger Euclidean-continuity on $\mathcal{L}$ (see, e.g., Abdellaoui 2002, Lemma 18). Further, the three conditions taken together imply the existence of a continuous function $V : \mathcal{L} \to \mathbb{R}$, strictly increasing in each cumulative probability, that represents $\succcurlyeq$.[8] The latter follows from results of Debreu (1954).

Next, we focus on the additive separability property across outcomes for the representing function V. This requires an independence condition for common cumulative probabilities. To define this property we introduce some useful notation. For $i \in \{1, \ldots, n-1\}$, $P \in \mathcal{L}$ and $\sigma \in [0,1]$, we denote by $\sigma_i P$ the prospect that agrees with P except that p_i is replaced by σ. Whenever this notation is used it is implicitly assumed that $p_{i-1} \leq \sigma \leq p_{i+1}$ (respectively, $\sigma \leq p_{i+1}$ if $i = 1$ and $p_{i-1} \leq \sigma$ if $i = n-1$) to ensure that $\sigma_i P \in \mathcal{L}$. More generally, for some nonempty subset $I \subset \{1, \ldots, n-1\}$ we write $\sigma_I P$ for the prospect P with p_i replaced by $\sigma \in [0,1]$ for all $i \in I$. Clearly, for $\sigma_I P$ to be a well-defined prospect the set I must include all indices between and including the smallest ($\min\{i : i \in I\}$) and the largest ($\max\{i : i \in I\}$) in the set I.

Independence: The preference relation $\succcurlyeq$ satisfies *independence of common cumulative probabilities* if $\sigma_i P \succcurlyeq \sigma_i Q \Leftrightarrow \rho_i P \succcurlyeq \rho_i Q$ for all $\sigma_i P, \sigma_i Q, \rho_i P, \rho_i Q \in \mathcal{L}$.

The next lemma follows from results of Wakker (1993) on additive representations on rank-ordered sets.

Lemma 1 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{L}$:*

[7] The ρ-probability mixture of P with R is the prospect $\rho P + (1-\rho)R = (\rho p_1 + (1-\rho)r_1, \ldots, \rho p_n + (1-\rho)r_n)$.
[8] This function may be unbounded at x_n or x_1.

(i) The preference relation $\succcurlyeq$ on $\mathcal{L}$ is represented by an additive function

$$V(P) = \sum_{j=1}^{n-1} V_j(p_j),$$

with continuous strictly increasing functions $V_1, \ldots, V_{n-1} : [0,1] \to \mathbb{R}$ which are bounded except maybe V_1 and V_{n-1} which could be unbounded at extreme probabilities (i.e., V_1 may be unbounded at 0 and V_{n-1} may be unbounded at 1).

(ii) The preference relation $\succcurlyeq$ is a Jensen-continuous weak order that satisfies first order stochastic dominance and independence of common cumulative probabilities.

The functions $V_1, \ldots, V_{n-1}$ are jointly cardinal, that is, they are unique up to multiplication by a common positive constant and addition of a real number. $\square$

Next we focus on the condition that, if added to Lemma 1, delivers PT. We present this principle in the next Section.

3 Consistent Probability Midpoints

In this section we present consistency requirements for elicited probability midpoints. These requirements will depend only on the sign of the outcomes attached to these midpoints. This way sign-dependence, a novelty introduced by PT, is accounted for. Our main theorem shows that, when added to statement (ii) of Lemma 1, our consistency principle for elicited midpoints indeed delivers PT.

To motivate our consistency property, suppose that we have $I = \{1, \ldots, i\}$ with $i < k$ and for some $\alpha, \beta \in [0,1]$, $\alpha < \beta$, we observe the indifference $\alpha_I P \sim \beta_I Q$. Such indifferences can be elicited as indicated in Figure 1 below, by asking subjects to indicate or reveal the probability that makes them indifferent between the prospect in Panel (a), where x_P indicates that residual probability $(1 - \alpha)$ is given to outcomes $x_j, j \notin I$ as specified by P, and the prospect in Panel (b).

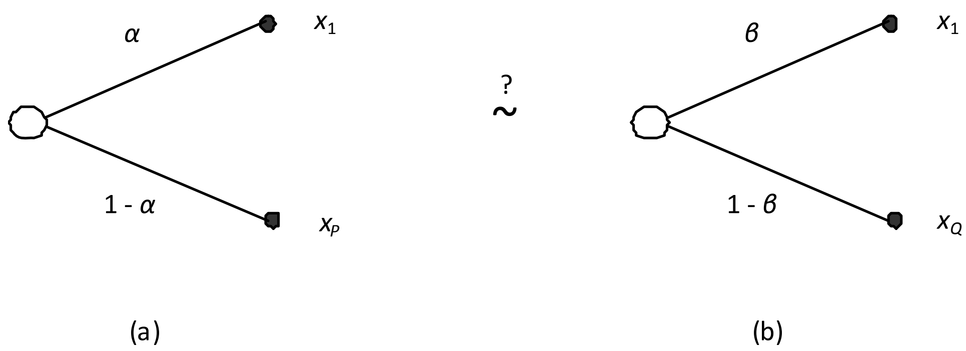

Figure 1: Eliciting probability β to give indifference $\alpha_I P \sim \beta_I Q$.

Such elicitation methods work well for utility measurement (e.g., Baillon, Driessen and Wakker 2012, Abdellaoui, Attema and Bleichrodt 2010) for measuring discounting functions for time preferences (Attema, et al. 2010), and have been adopted for measurements of weighting functions under risk and uncertainty (Abdellaoui, et al. 2011, Abdellaoui, Diecidue and Onçuler 2011). These methods can also play an important role in decomposing weighting functions under uncertainty into a probabilistic risk component and a pure belief component as suggested in Wakker (2004).

Notice that all outcomes $x_j, j \in I$, in the preceding prospects are gains. Suppose that for the prospect $\alpha_I P$ we improve the likelihood of the gain x_1 by $\beta - \alpha$, resulting in the prospect $\beta_I P$. In that case, for the prospect $\beta_I Q$ we require an improvement in the likelihood of outcomes x_1 to restore the preceding indifference. Suppose that $\gamma - \beta$ is required to maintain the indifference, such that we obtain the new indifference $\beta_I P \sim \gamma_I Q$. Both improvements in the probability of outcome x_1 come at the expense of reducing the (cumulative) probability of outcome x_{i+1} by the corresponding probability mass. Condition $i < k$ ensures that x_{i+1} is not a loss; x_{i+1} is a gain or can be the reference point. This means that the probabilities of losses remain unaffected by these joint improvements, and in general they can be chosen equal in P and Q, such that, according to independence of common cumulative probabilities, they should be of no influence for the derived indifferences.

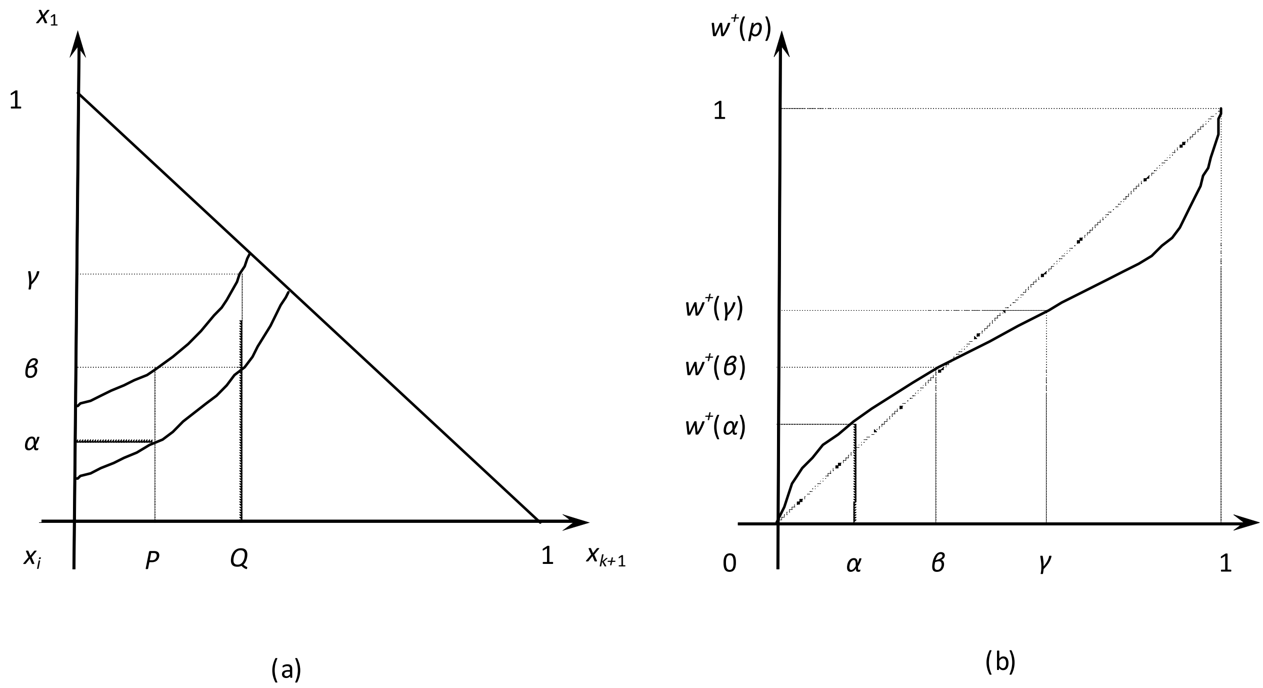

Figure 2: Elicited probability midpoints (a) and (b) perceived impact on w^+.

In Figure 2 (a) above, the original indifference before and after the improvements are depicted in a probability triangle. This means that the improvement $\beta - \alpha$ in the cumulative probability of prospect $\alpha_I P$ is of similar weight (or treated the same) as the improvement $\gamma - \beta$ in the cumulative probability of prospect $\beta_I Q$.

Panel (b) of Figure 2 shows how these improvements are treated by the weighting function for gain probabilities. Substitution of PT from Equation (3) into the indifferences $\alpha_I P \sim \beta_I Q$ and $\beta_I P \sim \gamma_I Q$, and taking the difference of the resulting equations implies, after elimination of common terms, that

$$[w^+(\alpha) - w^+(\beta)]u(x_1) = [w^+(\beta) - w^+(\gamma)]u(x_1).$$

As $x_1 \succ x_k$ we have $u(x_1) > 0$, which is common on both sides of the preceding equation and can, therefore. be eliminated. We obtain

$$w^+(\alpha) - w^+(\beta) = w^+(\beta) - w^+(\gamma),$$

or

$$w^{+}(\beta) = \frac{w^{+}(\gamma) - w^{+}(\alpha)}{2}. \tag{4}$$

This shows that, under PT, β is perceived as the probability *midpoint* between α and γ on the w^{+}-scale. Moreover, Equation (4) is independent of any gains or the utility thereof, which suggests that the joint improvements $\beta - \alpha$ and $\gamma - \beta$ should be treated the same irrespective of the common gain where the improvement is made. For example if the cumulative probability of the gain x_2 is improved in both prospects we should observe $\alpha_1 \beta_{I\setminus\{1\}} P \sim \beta_1 \gamma_{I\setminus\{1\}} Q$. Indeed one can verify, similar to the preceding derivation, that substitution of PT into $\alpha_I P \sim \beta_I Q$ and $\alpha_1 \beta_{I\setminus\{1\}} P \sim \beta_1 \gamma_{I\setminus\{1\}} Q$ also implies Equation (4). The following consistency property for elicited probability midpoints is obtained.

Midpoint Consistency: The preference $\succcurlyeq$ satisfies *probability midpoint consistency* if $\alpha_I P \sim \beta_I Q$ and $\beta_I P \sim \gamma_I Q$ imply $\alpha_J \beta_{I\setminus J} P \sim \beta_J \gamma_{I\setminus J} Q$ whenever $J \subset I \subset \{1, \ldots, n-1\}$ such that either all $x_i, i \in I$, are gains or all $x_{i+1}, i \in I$, are losses.

Before we present the main result of the paper, we illustrate the analogous derivation of midpoints for the weighting function for loss probabilities. Suppose that we have $I = \{i, \ldots, n-1\}$ with $k \leq i$ and for some $\alpha, \beta \in [0,1]$, $\alpha < \beta$, we observe the indifference $\alpha_I P' \sim \beta_I Q'$. Then reducing the probability of obtaining x_n by $\beta - \alpha$ in prospect $\alpha_I P'$ gives the improved prospect $\beta_I P'$. This improvement comes as a result of simultaneously increasing the likelihood of outcome x_i by $\beta - \alpha$. To reinstate indifference after this improvement we need to reduce in prospect $\beta_I Q'$ the probability of x_n, by say $\delta - \beta$. We obtain $\beta_I P' \sim \delta_I Q'$. Then substitution of PT for $\alpha_I P' \sim \beta_I Q'$ and for $\beta_I P' \sim \delta_I Q'$ and taking the difference of the two equations gives, after cancellation of common terms, the following relationship:

$$[w^{-}(1-\alpha) - w^{-}(1-\beta)]u(x_n) = [w^{-}(1-\beta) - w^{-}(1-\delta)]u(x_n).$$

Given that $k \leq i < n$ and $x_k \prec x_n$ we have $0 > u(x_n)$. We eliminate $u(x_n)$ from the preceding equation and obtain

$$w^{-}(1-\beta) = \frac{w^{-}(1-\delta) - w^{-}(1-\alpha)}{2}. \tag{5}$$

This means that, given $(1-\alpha)$ and $(1-\delta)$ for the weighting function for loss probabilities, $(1-\beta)$ is a probability midpoint. Reformulated in terms of the dual $\tilde{w}^{-}$ of w^{-} one obtains

$$\tilde{w}^{-}(\beta) = \frac{\tilde{w}^{-}(\delta) - \tilde{w}^{-}(\alpha)}{2}$$

confirming that β is a probability midpoint of α and δ for the dual of the weighting function for loss probabilities. Again we notice that the latter two equations are independent of the utilities of the losses that were attached to the elicited probability midpoint if PT is assumed.

We can now formulate the main result of the paper.

THEOREM 2 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{L}$:*

(i) The preference relation $\succcurlyeq$ on $\mathcal{L}$ is represented by prospect theory with strictly increasing and continuous probability weighting functions w^{+} and w^{-}, and utility $u : X \to \mathbb{R}$ that agrees with the ordering of outcomes and assigns $u(x_k) = 0$ to the reference point x_k.

(ii) The preference relation $\succcurlyeq$ is a Jensen-continuous weak order that satisfies first order stochastic dominance, independence of common cumulative probabilities and probability midpoint consistency.

The probability weighting functions are uniquely determined and the utility function is a ratio scale. $\square$

4 Extensions

In the previous sections we have assumed that we have strictly ordered outcomes and at least two gains and two losses in addition to the reference point. First, the strict ordering can be relaxed if there are at least five strictly ordered outcomes, two gains, two losses in addition to the reference point. If X is finite all results remain valid if we take representatives for each set of indifferent outcomes. These outcomes will then be given the same utility value. If, however, X is infinite, then results remain valid for each finite subset of outcomes Y that contains at least two gains, two losses and the reference point all of which are strictly ordered. We can then extend the PT-representations on the sets of prospects over the different finite subsets to a general PT-representation by using the fact that the representations on any such sets of prospects over Y and of prospects over Y' must agree with the representation on the set of prospects over $Y \cup Y'$. Hence a common PT-representation must exist over prospects over the possibly infinite X.

Second, the requirement of having at least five strictly ordered outcomes can be relaxed to having at least four strictly ordered outcomes. If X contains only gains and possibly the reference outcome (or only losses and possibly the reference outcome), then the probability midpoint consistency condition comes down to the consistency in probability attitudes principle of Zank (2010). In that case PT reduces to RDU and we can apply the results of Zank (2010) to obtain a corresponding preference foundation.

If there are only three strictly ordered outcomes, the probability midpoint consistency principle is void. In that case, we can still obtain RDU if we do not have outcomes of opposite signs (i.e., all outcomes are the reference outcome or gains or all outcomes are losses or the reference outcome) by invoking the probability tradeoff consistency principle of Abdellaoui (2002) or a refinement of that principle as proposed in Köbberling and Wakker (2003). In that case one can also drop independence of common cumulative probabilities as it is implied by the stronger probability tradeoff consistency principle. If, however, we have one gain, the reference outcome and one loss, then Lemma 1 no longer holds as independence of common cumulative probabilities is insufficient to deliver additive separability. We can still derive an additive representation by using stronger conditions like the Thomsen condition or triple cancellation as in Wakker (1993, Theorem 3.2). Then, if those additive functions are finite at extreme probabilities, they can be seen as the product of utility times the corresponding weighting function, and we immediately obtain PT. For fewer than three strictly ordered outcomes first stochastic dominance and weak order are suficient for an ordinal representation of preferences.

In our derivation of PT it has been essential that the weighting functions are continuous at 0 and at 1. Discontinuities at these extreme probabilities are, however, empirically meaningful. We could adopt a weaker version of Jensen-continuity that is restricted to prospects that have common best and worst outcomes with positive objective probability. Such conditions have been used in Cohen (1992) and more recently in Webb and Zank (2011) where probability weighting functions that are linear derived that are linear and discontinuous at extreme probabilities. These weighting functions can then be described by two parameters one for optimism and one for pessimism. As Webb and Zank show, this relaxation of continuity in probabilities comes at a price. They require additional structural assumptions for the preference in order to obtain consistency of the parameters across sets of prospects with different minimal and maximal outcomes. Also specific consistency principles that imply the uniqueness of these parameters are required. We conjecture that in our framework such consistency principles can be formulated for nonlinear weighting functions that are discontinuous at 0 and at 1. A formal derivation of PT with such weighting functions is, however, beyond the scope of this paper.

5 Conclusion

Prospect theory is currently the most popular descriptive theory for decision under risk and uncertainty. PT incorporates several prominent behavioral phenomena and can explain many empirical regularities that influence risk attitudes. Loss aversion and diminishing sensitivity of utility are seen as features of a continuous utility function (Kahneman and Tversky 1979, Tversky and Kahneman 1992, Tversky and Wakker 1993, Köbberling and Wakker 2005) and this may explain why all existing foundations of prospect theory have accordingly provided preference conditions that rely on the existence of a continuous utility. The focus of this paper has been on sign-dependence, the different treatment of probabilities depending on the whether the latter are attached to gains or to losses. We have complemented the existing foundations for PT in the "continuous utility approach" with preference foundations based on the "continuous weighting function approach" by adopting a familiar tool from empirical measurement of probability weighting functions and have demonstrated how PT can be obtained in an efficient and tractable manner from behavioral principles.

Appendix: Proofs

Proof of Lemma 1: The proof of the lemma follows from results for additive representations on rank-ordered sets in Wakker (1993, Theorem 3.2 and Corollary 3.6). That statement (i) implies statement (ii) is immediate from the properties of the functions $V_j, j = \{1, \ldots, n-1\}$. As we have a preference relation $\succcurlyeq$ defined on a rank-ordered set of cumulative probabilities (i.e., a rank-ordered subset of $[0,1]^{n-1}$) and $\succcurlyeq$ satisfies weak order, Jensen-continuity and first order stochastic dominance, we also have Euclidean continuity (by Lemma 18 in Abdellaoui 2002) for $\succcurlyeq$. First order stochastic dominance comes down to strong monotonicity in cumulative probabilities. Further, as $n > 4$ we can use independence of common cumulative probabilities, which comes down to coordinate independence of Wakker (1993), and note that statement (ii) of Theorem 3.2 of Wakker is satisfied. Then statement (i) of the lemma follows from statement (i) of Theorem 3.2 of Wakker, the only difference being that our strong monotonicity implies that the functions $V_j, j = \{1, \ldots, n-1\}$ are strictly increasing. Uniqueness results are as in Wakker's Theorem 3.2. This concludes the proof of Lemma 1. $\square$

Proof of Theorem 2: The derivation of statement (ii) from statement (i) follows from Lemma 1 and the analysis preceding the theorem in the main text on the consistency of elicited probability midpoints under PT.

We now prove that statement (ii) implies statement (i) of the theorem. Assume that $\succcurlyeq$ on $\mathcal{L}$

is a weak order that satisfies first order stochastic dominance, independence of common cumulative probabilities and probability midpoint consistency. Then, by statement (i) of Lemma 1 the preference $\succcurlyeq$ on $\mathcal{L}$ is represented by an additive function

$$V(P) = \sum_{j=1}^{n-1} V_j(p_j), \tag{6}$$

with continuous strictly increasing functions $V_1, \ldots, V_{n-1} : [0,1] \to \mathbb{R}$ which are bounded except maybe V_1 and V_{n-1} which could be unbounded at extreme probabilities.

Next we restrict our analysis to cumulative probabilities different from 0 or 1 in order to avoid the problems with the unboundedness of V_1 and V_{n-1}. For any $\delta \in (0,1)$ and $\varepsilon > 0$ let $B_\varepsilon(\delta)$ be the open neighborhood around δ with Euclidean distance ε. Take any $\alpha, \beta, \gamma \in B_\varepsilon(\delta)$ such that

$$\sum_{i=1}^{k-1} [V_i(\beta) - V_i(\alpha)] = \sum_{i=1}^{k-1} [V_i(\gamma) - V_i(\beta)]. \tag{7}$$

Then, for $\varepsilon > 0$ sufficiently small, by continuity of the functions $V_i, i = k, \ldots, n-1$, there exists lotteries $P, Q \in \mathcal{L}$ with

$$\sum_{i=1}^{k-1} V_i(\alpha) + \sum_{i=k}^{n-1} V_i(p_i) = \sum_{i=1}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(q_i)$$

and

$$\sum_{i=1}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(p_i) = \sum_{i=1}^{k-1} V_i(\gamma) + \sum_{i=k}^{n-1} V_i(q_i).$$

The latter two equations are equivalent to the respective indifferences

$$\alpha_I P \sim \beta_I Q \text{ and } \beta_I P \sim \gamma_I Q,$$

where $I = \{1, \ldots, k-1\}$, meaning that the cumulative probabilities α, β, γ are attached to gains. Consider the case $\alpha < \beta$ (and note that the case $\alpha > \beta$ is completely analogous). By first order stochastic dominance it follows that $\gamma > \beta$. Further, probability midpoint consistency requires that

$$\alpha_J \beta_{I\setminus J} P \sim \beta_J \gamma_{I\setminus J} Q$$

for all $J = \{1, \ldots, j\}, j \in I\setminus\{k-1\}$. First take $j = 1$. Then, substitution of Equation (6) into $\alpha_I P \sim \beta_I Q$ implies

$$\sum_{i=1}^{k-1} V_i(\alpha) + \sum_{i=k}^{n-1} V_i(p_i) = \sum_{i=1}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(q_i),$$

and substitution of Equation (6) into $\alpha_1 \beta_{I\setminus\{1\}} P \sim \beta_1 \gamma_{I\setminus\{1\}} Q$ gives

$$V_1(\alpha) + \sum_{i=2}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(p_i) = V_1(\beta) + \sum_{i=2}^{k-1} V_i(\gamma) + \sum_{i=k}^{n-1} V_i(q_i).$$

Taking the difference of the two latter equations and cancelling common terms implies

$$\sum_{i=2}^{k-1} [V_i(\beta) - V_i(\alpha)] = \sum_{i=2}^{k-1} [V_i(\gamma) - V_i(\beta)].$$

Similarly, joint substitution of Equation (6) into $\beta_I P \sim \gamma_I Q$ and $\alpha_1 \beta_{I\setminus\{1\}} P \sim \beta_1 \gamma_{I\setminus\{1\}} Q$, taking differences and cancelling common terms, implies

$$V_1(\beta) - V_1(\alpha) = V_1(\gamma) - V_1(\beta). \tag{8}$$

Similarly, if $j = 2$, we obtain

$$\sum_{i=3}^{k-1} [V_i(\beta) - V_i(\alpha)] = \sum_{i=3}^{k-1} [V_i(\gamma) - V_i(\beta)]$$

and

$$\sum_{i=1}^{2} [V_i(\beta) - V_i(\alpha)] = \sum_{i=1}^{2} [V_i(\gamma) - V_i(\beta)],$$

and using Equation (8) we obtain

$$V_2(\beta) - V_2(\alpha) = V_2(\gamma) - V_2(\beta).$$

By induction on j we conclude that if Equation (7) holds then for all $j = 1, \ldots, k-1$ we have

$$V_j(\beta) - V_j(\alpha) = V_j(\gamma) - V_j(\beta).$$

That the converse holds is immediate. We conclude that for any $\delta \in (0,1)$ and sufficiently small $\varepsilon > 0$ for $\alpha, \beta, \gamma \in B_\varepsilon(\delta)$ we have

$$\begin{aligned} \sum_{i=1}^{k-1}[V_i(\beta) - V_i(\alpha)] &= \sum_{i=1}^{k-1}[V_i(\gamma) - V_i(\beta)] \\ &\Leftrightarrow \\ V_j(\beta) - V_j(\alpha) &= V_j(\gamma) - V_j(\beta) \text{ for all } j = 1, \ldots, k-1. \end{aligned}$$

This means that locally the functions V_j, $j = 1, \ldots, k-1$, are proportional and also proportional to their sum, which we denote V^+. From local proportionality and continuity global proportionality follows. This means that there exist positive constants $s_1, \ldots, s_{k-1}$ and real numbers $t_1, \ldots, t_{k-1}$ such that

$$V_j(\cdot) = s_j V^+(\cdot) + t_j, j = 1, \ldots, k-1.$$

Following Proposition 3.5 of Wakker (1993) the functions V_j can be taken finite at 0 and 1, and can continuously be extended to all of $[0,1]$.

Arguments that are similar to those used to derive proportionality of the functions V^+ and V_j, $j = 1, \ldots, k-1$, now applying midpoint consistency for probability midpoints of losses, are used to derive proportionality of the functions $V^- := \sum_{j=k}^{n-1} V_j$ and V_j, $j = k, \ldots, n-1$. Proposition 3.5 of Wakker (1993) applies again saying that the functions V_j can be taken finite at 0 and 1, and can continuously be extended to all of $[0,1]$. We conclude that here exist positive constants $s_k, \ldots, s_{n-1}$ and real numbers $t_k, \ldots, t_{n-1}$ such that

$$V_j(\cdot) = s_j V^-(\cdot) + t_j, j = k, \ldots, n-1.$$

Next, we derive the weighting functions for probabilities of gains and losses and the utility for outcomes. We fix $V^+(1) + V^-(1) = 1$ and $V_j(0) = 0$ for $j = 1, \ldots, k-1$ and $V_j(1) = 0$ for $j = k, \ldots, n-1$, thereby fixing the scale and location of the otherwise jointly cardinal functions V_j. Then, $t_1 = \cdots = t_{n-1} = 0$ must hold and it follows that $V^+(1) = 1$. We define

$$w^+(p) := V^+(p) = \sum_{j=1}^{k-1} V_j(p) + \sum_{j=k}^{n-1} V_j(1).$$

Therefore, $w^+(0) = 0, w^+(1) = 1$ and w^+ is strictly increasing and continuous on $[0,1]$, and hence, indeed a well-defined probability weighting function. It is the probability weighting function for probabilities of gains.

Next we derive w^-. First we define

$$\hat{w}(p) := \frac{V^-(p)}{V^-(0)} = \frac{\sum_{j=1}^{k-1} V_j(0) + \sum_{j=k}^{n-1} V_j(p)}{\sum_{j=1}^{k-1} V_j(0) + \sum_{j=k}^{n-1} V_j(1)}.$$

This is a well-defined function given that the functions $V_j, j = k, \ldots, n-1$, are strictly increasing and bounded, and thus, $V_j(p) < 0$ for all $j = k, \ldots, n-1$, whenever $p < 1$, such that the denominator $\sum_{j=1}^{k-1} V_j(0) + \sum_{j=k}^{n-1} V_j(1) \neq 0$ and finite. It then follows that the function $\hat{w}$ has the following properties: $\hat{w}(1) = 0$ and $\hat{w}(0) = 1$ and $\hat{w}$ is strictly decreasing and continuous on $[0,1]$. We set

$$\tilde{w}^-(p) := 1 - \hat{w}(p) = \frac{V^-(0) - V^-(p)}{V^-(0)},$$

for each $p \in [0,1]$, which gives us the dual weighting function for probabilities of losses. A useful rearrangement of this equation gives

$$V^-(p) = V^-(0)[1 - \tilde{w}^-(p)].$$

From $\tilde{w}^-$ we obtain w^- through $w^-(p) = 1 - \tilde{w}^-(1-p)$ for all $p \in [0,1]$.

Next we derive the utility function for outcomes. From the derivation of w^+ and $V_j(\cdot) = s_j V^+(\cdot), j = 1, \ldots, k-1$, we obtain

$$V_j(\cdot) = s_j w^+(\cdot), j = 1, \ldots, k-1,$$

and from the derivation of $\tilde{w}^-$ and $V_j(\cdot) = s_j V^-(\cdot), j = k, \ldots, n-1$, we obtain

$$V_j(\cdot) = s_j V^-(0)[1 - \tilde{w}^-(\cdot)], j = k, \ldots, n-1.$$

Noting that the degenerate lottery that gives x_i for sure is expressed as the prospect $0_{\{1,\ldots,i-1\}}(1, \ldots, 1)$, for each $i = 1, \ldots, n$, we define utility as follows:

$$\begin{aligned} u(x_k) &:= V(0_{\{1,\ldots,k-1\}}(1, \ldots, 1)) \\ &= V^-(1) \\ &= 0. \end{aligned}$$

Then, for $i = k-1, \ldots, 1$ we iteratively define

$$u(x_i) := u(x_{i+1}) + s_i.$$

And for $i = k+1, \ldots, n$ we iteratively define

$$u(x_i) := u(x_{i-1}) + s_{i-1} V^-(0).$$

This way defined the ordering of the utility for outcomes is $u(x_1) > \cdots > u(x_n)$, thus in agreement with the ordering according to the preference $\succcurlyeq$.

Next, substitution into $V(P)$, gives

$$\begin{aligned} V(P) &= \sum_{j=1}^{k-1} V_j(p_j) + \sum_{j=k}^{n-1} V_j(p_j) \\ &= \sum_{j=1}^{k-1} s_j w^+(p_j) + \sum_{j=k}^{n-1} s_j V^-(0)[1-\tilde{w}^-(p_j)]. \end{aligned}$$

Note that for $i = 1, \ldots, k-1$ we have $s_i = u(x_i) - u(x_{i+1})$ and for $i = k, \ldots, n-1$ we have $s_i V^-(0) = u(x_{i+1}) - u(x_i)$. Substitution into the preceding equation gives

$$\begin{aligned} V(P) &= \sum_{j=1}^{k-1} w^+(p_j)[u(x_j) - u(x_{j+1})] + \sum_{j=k}^{n-1}[u(x_{j+1}) - u(x_j)][1-\tilde{w}^-(p_j)] \\ &= \sum_{j=1}^{k-1}[w^+(p_j) - w^+(p_{j-1})]u(x_j) + \sum_{j=k}^{n-1}[u(x_{j+1}) - u(x_j)][1-\tilde{w}^-(p_j)], \end{aligned}$$

where, in the latter equation, the term relating to probabilities of gains has been rearranged using the properties that $w^+(p_{j-1}) = 0$ for $j = 1$ and $u(x_{j+1}) = 0$ for $j = k-1$. Next we rearrange the term relating to probabilities of losses. After substitution of $\tilde{w}^-(p) = 1 - w^-(1-p)$, we obtain

$$\begin{aligned} V(P) &= \sum_{j=1}^{k-1}[w^+(p_j) - w^+(p_{j-1})]u(x_j) + \sum_{j=k}^{n-1}[u(x_{j+1}) - u(x_j)]w^-(1-p_j) \\ &= \sum_{j=1}^{k-1}[w^+(p_j) - w^+(p_{j-1})]u(x_j) + \sum_{j=k+1}^{n}[u(x_j) - u(x_{j-1})]w^-(1-p_{j-1}). \end{aligned}$$

Rearranging we obtain

$$V(P) = \sum_{j=1}^{k-1}[w^+(p_j) - w^+(p_{j-1})]u(x_j) + \sum_{j=k+1}^{n}[w^-(1-p_{j-1}) - w^-(1-p_j)]u(x_j) = PT(P), \tag{9}$$

where, in the derivation latter expression for loss probabilities, we have used that $u(x_{j-1}) = 0$ for

$j = k+1$ and $w^-(1-p_j) = 0$ for $j = n$ (recall that $p_n = 1$). We conclude that the representation V of $\succcurlyeq$ on $\mathcal{L}$ is, in fact, a PT-functional. Hence, PT represents $\succcurlyeq$ on $\mathcal{L}$. This concludes the derivation of statement (i) from statement (ii) in Theorem 2.

To complete the proof of the theorem we need to derive uniqueness results. In the derivation of w^+ and w^- we have fixed the scale and location of the otherwise jointly cardinal functions $V_j, j = 1, \ldots, n-1$. That is, given any other representation of preferences that is additively separable as in Lemma 1, fixing scale and location as required in the proof above will lead to the same probability weighting functions. This shows that the weighting functions w^+ and w^- are uniquely determined. From the definition of the utility function u it is clear that the only freedom that we have in defining utility is the starting value at the reference point x_k (i.e., the location parameter) and a scaling parameter due to the jointly cardinal functions $V_j, j = 1, \ldots, n-1$. So, utility can, at most, be cardinal. However, in order to rewrite V in form of the PT-functional it is critical that $u(x_k) = 0$. Otherwise, if $u(x_k) \neq 0$ the terms $w^+(p_k)u(x_k)$ and $u(x_k)w^-(1-p_k)$ will appear in Equation (9). With these terms added in Equation (9) a functional is obtained that violates first order stochastic dominance and continuity, hence, cannot be a representation of $\succcurlyeq$ on $\mathcal{L}$. This means that u must be a ratio scale.

This concludes the proof of Theorem 2. $\square$

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