A New Approach to Correlation of Types in Bayesian Games*

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Abstract

Despite their importance, games with incomplete information and dependent types are poorly understood; only special cases have been considered and a general approach is not yet available. In this paper, we propose a new approach to the model of correlation of types in Bayesian games, which also allows asymmetries. This is related to the idea that "beliefs do not determine preferences," and consists of modeling types with two explicit parts: one for preferences and another for beliefs. Building on this idea, we are able to provide the first pure strategy equilibrium existence for a general model of multi-unit auctions where types can be correlated. We also provide further results for a particular case of this idea, which we call "very simple distributions." These distributions are defined by density functions which are constant in squares covering the support of all types. We provide necessary and sufficient conditions for the existence of a symmetric monotonic pure strategy equilibrium in first-price auctions with these distributions.

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1 Introduction

Correlation is a central phenomenon in our world. It is present not only among many variables relevant to the real economy, but it is probably important in virtually all economic fields. And, yet, we have not been able to properly tackle correlation. Simply put, it is a difficult subject that has been defying our standard techniques.

A particular field where correlation has been poorly understood is the subject of games of incomplete information. To begin with, we are unable even to establish general pure strategy equilibrium existence results when types are not independent. For instance, the important papers of Reny (1999), McAdams (2003) and Reny (2011), which continue the work of Athey (2001), establish equilibrium existence for multi-unit auctions only in the case of independence.

After the pioneering work of Wilson (1969) and (1977), one of the most important contribution to the study of dependence in games was made by Milgrom and Weber (1982) when they introduced affiliation. This became a central assumption in the study of correlated types and it is, to this date, a hallmark of economic theory. However, some of the implications of affiliation are not robust to other forms of dependence—see de Castro (2011).

Even with affiliation, however, progress in the study of Bayesian games with correlated types has been slow and difficult.[1] The case of first price auction with interdependent values with two players was obtained by Athey (2001) and Lizzeri and Persico (2000). Later, Reny and Zamir (2004) extended this to n players. In the case of multi-dimensional auctions, pure strategy equilibrium existence (PSEE) is established exclusively for independent types; Reny (1999), McAdams (2003), Jackson and Swinkels (2005), Reny (2011). There is not a single general result of PSEE with affiliation for multi-unit auctions. Indeed, McAdams (2007) provides a counterexample for the PSEE in a uniform price auction with affiliated types. Departing from the question of equilibrium existence, affiliation is also not enough to imply the linkage principle in multi-unit auctions (see Perry and Reny (1999)). This suggests a difficult tension: it seems that we need stronger conditions, but at the same time we would like to have a more flexible framework. But what do we know outside of affiliation?

The more general papers that allow correlation of types—Jackson, Si-

[1] We will just highlight some of the standing gaps in the literature. For a more complete survey, see de Castro and Karney (2011).

mon, Swinkels, and Zame (2002) or Jackson and Swinkels (2005)—prove equilibrium existence only in mixed strategies. The question is whether pure strategy equilibria exist. Some general results have been established by van Zandt and Vives (2007) and van Zandt (2010), but these papers require assumptions that are not valid for some important classes of Bayesian games.[2] Out of affiliation, we do not have general pure strategy equilibria existence results even for games as simple as symmetric private value first price auctions.[3] Thus, PSEE in Bayesian games out of independent types is an old, but still open question.

The above discussion should not give the impression that problems are restricted to equilibrium existence. Rather, it is just an illustration of how deep the limitations go. Since equilibrium existence is the basis for all useful results that game theory is able to provide, these limitations hinder progress in econometrics, industrial organization and many other practical applications of Bayesian games.

The purpose of this paper is to present a simple idea and suggest it as a different route to tackle the problem of correlation of types in Bayesian games. To explain the idea, it is useful to recall the "standard approach" to model correlation. This "standard approach" or "setting" is the representation of players' information by signals and a common prior determining the beliefs about other types through conditional probabilities. That is, in a standard setting, each player i has a signal v_i that represents the preference parameter for that player (e.g., the value of an object), there is a common prior γ over the signals of all players $v = (v_1, \ldots, v_n)$ and a player with signal v_i has belief δ_i given by the conditional probability $\gamma(\cdot|v_i)$. In this sense, the signal contains both information about the preferences (e.g. the value of the object) and the beliefs about other players' information. Although nowadays collapsing players' beliefs and tastes in this way seems very natural, when Harsanyi (1967-8) introduced the idea of types, he was careful to maintain different parameters for tastes and beliefs. The standard approach became very popular later, and its main justification seems to be its simplicity.[4] A surprising implication of our results is that the theory becomes considerably simpler if we *do not* make this simplifying assumption.

[2]The utility functions are required to have increasing differences in all actions, a property that does not hold in auctions, for instance.

[3]de Castro (2007) and Monteiro and Moreira (2006) present partial results in this particular setting.

[4]Milgrom and Weber (1982, Footnote 14, p. 1097) justify the standard approach as follows: "To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the

We begin by departing from this "standard approach" and explicitly describe each type t_i as composed of two separate parts: a preference parameter v_i and a belief parameter δ_i, that is, $t_i = (v_i, \delta_i)$.[5] The important point is that beliefs and preference parameters should be kept as separate elements.[6] Of course, this simple idea requires more details. The main contribution of this paper is the introduction of the following condition:[7]

$(*)$ Any non-null set of types contains two strictly ordered types sharing the same belief.

Everything that follows in this paper is either a discussion of what $(*)$ means or an illustration of the results that it allows us to prove, together with some technical contributions that simplify or generalize previously known settings and results.

First, observe that the most important aspect in condition $(*)$ is the fact that types share the same belief. Indeed, the order implicitly assumed to exist in $(*)$ can be constructed from the preference part of each type. For instance, if the preference part V_i refers to the values of the objects in a multi-unit auction, we could order its elements v_i using the standard coordinate-wise order of euclidean spaces. Then, any set with positive measure will always contain two strictly ordered signals—see details in section 2.1. In this sense, the belief part is the most important restriction and we will focus the discussion here on this aspect. Let us see some settings where $(*)$ is satisfied:

1. Standard setting with Independence. Indeed, if types are independent, then any signals v_i, v_i' imply the same (conditional probability

information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders. The derivation of such a statistic from several separate pieces of information is in general a difficult task (...). It is in the light of these difficulties that we choose to view each X_i as a "value estimate," which may be correlated with the "estimates" of others but is the only piece of information available to bidder i."

[5] From a general perspective—that of universal type spaces introduced by Mertens and Zamir (1985)—, types can always be seen as having two parts: a payoff type and a belief type. Therefore, the explicit consideration of the belief goes in the direction of a more general model. This paper shows that this extra "complication" actually simplifies the analysis in a surprising way.

[6] It seems that Neeman (2004) was the first to argue that beliefs and preferences should be seen as "causally independent" from one another. As we discuss in more detail in section 7.1, our idea is intrinsically related to his "beliefs do not determine preferences" assumption. See also Heifetz and Neeman (2006).

[7] The condition implicitly requires that types spaces are ordered, as we further discuss below.

or) belief $\delta_i = \gamma(\cdot|v_i) = \gamma(\cdot|v_i')$.

2. Standard setting where there is only a countable number of different beliefs $\delta_i = \gamma(\cdot|v_i)$. Notice that independence is just a special case of this, where there is just *one* belief.

3. T_i is (a subset of) $V_i \times \Delta(T_{-i})$ and the measure on T_i is absolutely continuous with respect to the product of its marginals over V_i and $\Delta(T_{-i})$.

These examples will be fully specified and justified in section 2.1.

Of course, it is also useful to know settings where $(*)$ is not satisfied. A simple example is a standard setting with a uniform distribution γ over the triangle defined by $0 \leqslant v_1 \leqslant v_2 \leqslant 1$. In this case, if player 1 has signal v_1, her belief about player's 2 signal is the uniform distribution over $[v_1, 1]$. Therefore, $(*)$ cannot be satisfied because there are not two different signals sharing the same belief. Fortunately, however, for every standard setting where $(*)$ is not satisfied, there exists another setting sufficiently "close" that satisfies $(*)$. Indeed, assume that with probability $1 - \varepsilon$, the types and beliefs are just as described and, with probability $\varepsilon > 0$, when receiving the signal v_i, player i beliefs that other players' signals are uniformly distributed on $[0, 1]$. As we further discuss in section 2, this is "ε-close" to the original model and satisfies $(*)$. Therefore, $(*)$ does not add any significative extra restriction than that already imposed by the game model itself.

Having clarified some aspects about $(*)$, it is time to describe what $(*)$ allows us to accomplish. In the remainder of this introduction I describe the main results in this paper and other technical contributions, which could be of interest by themselves.

1.1 Main Results

The main results of the paper can be summarized as follows.

1. For Bayesian games with (possibly infinite dimensional) action spaces satisfying some weak assumptions (details in section 3), a generalization of supermodularity and increasing differences and general type spaces satisfying $(*)$, we show that every best reply to any mixed strategies is pure. That is, we give conditions under which strictly mixed strategies are never best replies. Therefore, *all* equilibria are in pure strategies. See Theorem 4.3 in section 4.

2. Building on this first result, we turn to a more particular setting: that of multi-unit auctions studied by Jackson and Swinkels (2005). In this setting, we show that if $(*)$ is satisfied, there exists a monotonic pure strategy equilibrium. This is the first result in the literature that establishes existence in pure strategies for general multi-unit auctions out of independence. See Theorem 5.1 in section 5.

3. Although the equilibrium mentioned in 2 above is in "monotonic" strategies, this monotonicity depends on a special order on types, which may be different from the standard one, for example in single-unit auctions. This raises the question whether the pure strategy equilibria shown to exist will also be in monotonic strategies taking in account the usual order on real-valued signals. Considering a symmetric first price auction in a still general setting satisfying $(*)$—see point 5 in section 1.2 below—, we establish *necessary and sufficient* conditions for the existence of a monotonic equilibrium. Moreover, since the setting is specially suitable for numerical simulations, we are able to establish an algorithm for checking when there is or is not an equilibrium. The algorithm is surprisingly fast. While the best known algorithms for finding mixed strategy equilibria in finite games run in exponential time, our algorithm requires only $O(k^2)$ manipulations in an auction with 2 players and k intervals. See Theorems 6.2 and 6.4. Section 6.3 shows how numerical methods could establish some results about the revenue ranking of first and second-price auctions with general kind of correlation. Moreover, we show in section 6.5 that results obtained in this framework satisfy approximate results in any standard settting.

As this list of results highlights, the results of this paper are not restricted to pure strategy equilibrium existence. In particular, it is not our main purpose to offer alternative methods for proving pure strategy equilibrium existence, as Reny (1999), Athey (2001), McAdams (2003) and Reny (2011) do. Rather, the main objective of the paper is to propose a new approach to the correlation of types in Bayesian games, establish its foundations and provide some venues for applications.

1.2 Other Contributions

Besides the ideas in the proofs of the above mentioned results, we introduce some technical contributions that could be of interest by themselves. These are the following:

1. We introduce a new monotonicity condition in section 3.1, which generalizes supermodularity and increasing differences (it is implied by them). This condition assumes that all spaces are only partially ordered sets, that is, the space of actions is not required to be a lattice, as supermodularity or quasi-supermodularity require. Called unimaginatively "monotonicity condition," it allows us to prove that all best-reply actions, even to mixed strategies, are increasing. It is presented in two forms—an increasing differences form, which is preserved under integration and it is very easy to check—and a single-crossing form, which is slightly more general. See definitions 3.1 and 3.3.

2. For some cases where our new monotonicity condition fails, we show that a simple perturbation technique can be used to approximate such games with games that do satisfy the condition. This technique is at the heart of the proof of pure strategy strategy equilibrium existence in the private value multi-unit auctions (Theorem 5.1).

3. In the context of multi-unit auctions, we define a condition on tie-breaking rules (Assumption 5.2) that allows us to prove the modularity of allocations (Proposition 5.4) and payments (Corollary 8.13). This condition is satisfied by a specific tie-breaking rule used by McAdams (2003) and Reny (2011) for uniform-price auctions, but includes other potentially relevant rules and allow general auction formats. These results are instrumental for our proof of Theorem 5.1.

4. We also introduce a condition (Assumption 3.2) about the interplay between the order and the metric of the action space that allows us to state our more general results for compact metric space, instead of just euclidean spaces. This condition is automatically satisfied in euclidean spaces with the usual coordinate-wise order, but also in many standard function spaces and other infinitely dimensional spaces. We are not aware of any similar condition being considered in the literature.

5. Finally, we introduce a special class of distributions, which we call "grid" distributions or "very simple" distributions. Here, it is enough to define very simple distributions in the simple case of single dimensional types and just two players. For this, assume that the signals support is $[0, 1]$ and divide this interval into k equal pieces. A very simple distribution is any distribution defined by a density function

which is constant in the squares thus formed. Figure 1 below illustrates this contruction for $k = 2$ divisions of $[0, 1]$. To see that this satisfies $(*)$, even without explicit mention to the beliefs, notice that once the player learns v_1, she also knows the interval where her value is, and every other signal in that interval shares exactly the same belief about the other bidder's valuation. Our results about the complete characterization of monotonic equilibrium existence in first price auctions (point 3 in section 1.1 above) is given in this setting. These distributions prove to be very convenient for theoretical and numerical manipulations, besides being dense in the set of all distributions—which allow approximation to any standard setting.

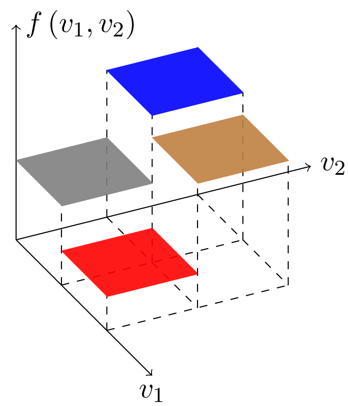

Figure 1: The density function of a grid distribution.

2 Framework for Types with Correlation

In this section we introduce our main idea: a model to study types with some form of dependence or correlation. The remainder of the paper will be devoted to explore the results that can be obtained within this framework.

Consider the set of player $I \equiv \{1, ..., N\}$. Eventually we will consider a non-strategic player, numbered 0, but this player does not need to have types and can be thought as of "Nature." Each individual $i \in I$ has a type $t_i \in T_i$, where $(T_i, \mathcal{T}_i)$ is a measurable space. Let $T \equiv \times_{i=1}^{N} T_i$ and $T_{-i} \equiv \times_{j \neq i} T_j$. The product σ-algebra on T and T_{-i} are denoted $\mathcal{T}$ and $\mathcal{T}_{-i}$, respectively. For a measurable space $(X, \mathcal{X})$, let $\Delta(X)$ denote the set of probability measures on $(X, \mathcal{X})$. Notice that we do not impose any topological assumptions in this setup.[8]

[8] Heifetz and Samet (1998) study type spaces without topology.

Player i's type determines the beliefs of player i about other players. This will be given by a map $\widehat{\delta}_i : T_i \to \Delta(T_{-i})$. Finally, we denote by $\Delta_i \subset \Delta(T_{-i})$ the set of player i's possible beliefs about other players' types, that is, $\Delta_i \equiv \widehat{\delta}_i(T_i)$.

For some results, it will be convenient to assume that the types $t_i \in T_i$ are generated by a measure τ_i. This is more natural when there is a common prior on T, although this is not strictly necessary. If there is a common prior on $(T, \mathcal{T})$, denoted τ, we will define τ_i as T_i-marginal of τ, that is:

$$\tau_i(B) \equiv \tau(T_1 \times ... \times T_{i-1} \times B \times T_{i+1} \times ... \times T_N), \forall B \in \mathcal{T}_i.$$

In this case, the belief map $\widehat{\delta}_i$ could be defined as a regular conditional probability of τ given t_i. Conditions for the existence of a regular conditional probability and relevant definitions can be found in the appendix. We emphasize, however, that the existence of a common prior is not strictly necessary.

Our main assumption requires us to consider a partial order on T_i. The condition requires that in any set of positive measure $E \subset T_i$, we can find a pair of types in such a set which are strictly ordered and have the same belief. In other words, the following repeats with symbols condition $(*)$ stated in the introduction.

> **Assumption** $(*)$**:** If $E \in \mathcal{T}_i$ has positive measure, i.e., $\tau_i(E) > 0$, then there exist $t_i, t_i' \in E$ such that $t_i <_i t_i'$ and $\widehat{\delta}_i(t_i) = \widehat{\delta}_i(t_i')$.

Note that $(*)$ is somewhat more demanding than "beliefs do not determine types," introduced by Neeman (2004). As Neeman's assumption, $(*)$ also requires that there are at least two types sharing the same belief, that is, $\widehat{\delta}_i(t_i) = \widehat{\delta}_i(t_i') = \delta$ above. However, it also requires that those types are ordered, while Neeman (2004) considers no order. Moreover, $(*)$ also requires we can find such a pair of ordered types in every set of positive measure. Note that the requirement that there are ordered types in a set of positive measure is not strong in continuous type spaces. In this setting, it just guarantees that the partial order is not too restricted.[9] On the other hand, although the existence of ordered pairs in any set of positive measure is, by itself, a weak requirement, $(*)$ requires in addition that this pair of types share the same belief. In section 2.1 below, we further specify the type space and describe a natural setting where $(*)$ holds.

[9] An example of a partial order that would fail that requirement is the equality order (two points are "ordered" only if they are equal). This is obviously a very restrictive partial order.

Note that this assumption rules out atomic (finite) type spaces. With atoms in the type spaces, it is well known that we can have only strictly mixed strategies as part of any equilibria, even in simple games. On the other hand, assuming that types are atomless is far from sufficient to guarantee equilibrium in pure strategies. For instance, the type spaces in Jackson and Swinkels (2005) are atomless, but they are able to prove equilibrium existence only in mixed strategies when some kind of correlation between the types is possible. Moreover, Radner and Rosenthal (1982) and Khan, Rath, and Sun (1999) provide examples of games with atomless types and without pure strategy equilibria.

2.1 Sufficient conditions and examples for $(*)$

It is perhaps useful to describe a more familiar setting where $(*)$ holds. For this, let $V = \times_{i=1}^{N} V_i$, where V_i is the part of the parameter space known to player i. For instance, V_i could represent the values of the objects to player i in a multi-unit private value auction. In this case, $V_i \subset \mathbb{R}_+^n$ has a natural order: the coordinate-wise order of euclidean spaces. Thus, we may assume that we have a natural partial order on V_i, which we will denote by $\succcurlyeq_i$. We can use this order to define a natural order on the type spaces as follows. Let $\widehat{V}_i : T_i \to V_i$ specify for each type $t_i \in T_i$, the preference parameter $\widehat{V}_i(t_i) \in V_i$ that is known by t_i. We can then define the order on T_i by:[10]

$$t_i' \geqslant_i t_i \iff \widehat{V}_i(t_i') \succcurlyeq_i \widehat{V}_i(t_i). \tag{1}$$

For all constructions in this section, we will assume the order on T_i defined by (1). We will consider two classes of examples: based on the standard approach, in which the beliefs are given by common priors, and settings where T_i is a (subset of) $V_i \times \Delta(T_{-i})$.

2.1.1 Sufficient conditions on the standard setting

As we described in the introduction, we call the standard approach one in which the types (or signals, actually) are given directly on $V = \times_{i=1}^{N} V_i$ and the beliefs about other players' signals are given by a joint distribution γ

[10] An alternative definition, sometimes convenient, is the following: $t_i' \geqslant_i t_i$ if and only if $\widehat{V}_i(t_i') \succcurlyeq_i \widehat{V}_i(t_i)$ and $\widehat{\delta}_i(t_i') = \widehat{\delta}_i(t_i)$, that is, we restrict (1) to be valid only for types with the same beliefs. All of our results remain valid without change under this alternative definition (some of them could be actually simplified). We use this alternative definition in our result about equilibrium existence on the JS auction. See (12).

on V. Of course, there is a σ-algebra Ξ such that (V, Ξ, γ) is a probability space. This space is the primitive in this model; we define a type space from this primitive as follows. Let T_i be defined as the following subset of $V_i \times \Delta(V_{-i})$:

$$T_i \equiv \{(v_i, \delta) \in V_i \times \Delta(V_{-i}) : \delta(\cdot) = \gamma(\cdot|v_i)\},$$

where $\gamma(\cdot|v_i)$ denotes, naturally, the γ-conditional probability on v_{-i} given v_i. Note that this T_i corresponds to the original signal space, described in the language of type of parameter spaces and beliefs. To complete the definition of the type space $(T_i, \mathcal{T}_i, \tau_i)$, let γ_i denote the marginal of γ on Θ_i,[11] and define: $\mathcal{T}_i \equiv \widehat{V}_i^{-1}(\Xi_i)$ and $\tau_i(E) \equiv \gamma_i(\widehat{V}_i(E))$, for every $E \in \mathcal{T}_i$.

To state our results, we need to introduce some notation. Let Δ_i denote the set of possible beliefs, that is,

$$\Delta_i \equiv \{\delta \in \Delta(V_{-i}) : \exists v_i \in V_i \text{ such that } (v_i, \delta) \in T_i\}.$$

Also, let $\Gamma : V_i \rightrightarrows \Delta_i$ be the correspondence defined by

$$\Gamma(v) \equiv \widehat{\delta}_i\left(\widehat{V}_i^{-1}(v)\right) = \{\delta \in \Delta_i : \exists t_i \in T_i \text{ s.t. } \widehat{V}_i(t_i) = v \text{ and } \widehat{\delta}_i(t_i) = \delta\}. \quad (2)$$

We first show that if there are a countable set of beliefs that "correspond" to almost all parameters, then $(*)$ holds if the parameters are "sufficiently" ordered. More precisely, we have the following:

Lemma 2.1 *The following conditions imply* $(*)$*:*

(i) For any measurable $E \subset V_i$, such that $\gamma_i(E) > 0$, there exist $v, v' \in E$ such that $v' \succ v$;[12]

(ii) There exists a countable set $\Delta_i' \subset \Delta_i$ such that for almost all $v \in V_i$, $\Gamma(v) \cap \Delta_i' \neq \emptyset$.

It is useful to discuss Lemma 2.1's conditions. Condition (i) is satisfied if the measure on V_i can be defined using sets of the form $[c, d] \equiv \{x \in T_i : c \leqslant_i x \leqslant_i d\}$. An example where this last condition holds is the following: T_i is an euclidean space, the Lebesgue measure is absolutely continuous with respect to τ_i and $\leqslant_i$ is the standard coordinate-wise order.[13]

[11]That is, $\gamma_i(E) \equiv \gamma(V_1 \times \cdots \times V_{i-1} \times E \times V_{i+1} \times \cdots \times V_N)$, for any measurable $E \subset V_i$.

[12]We write $x \succ y$ if $x \succcurlyeq y$ but not $y \succcurlyeq x$.

[13] By itself, this condition is weaker than Reny (2011)'s assumption G3, which requires that there is a countable subset T_i^0 of T_i such that every set in $\mathcal{T}_i$ assigned positive probability by τ_i contains two points between which lies a point in T_i^0. As Reny (2011, Lemma A.21, p. 546) shows, in the case T_i is a separable metric space, Reny's assumption G3 is equivalent to the requirement that every atomless set with positive measure contains two strictly ordered points, that is, Lemma 2.1's condition (i).

Condition (ii) is trivially satisfied if Δ_i is countable because $\Gamma(V_i) \subset \Delta_i$. In particular, if γ implies independent types, then Δ_i is unitary and condition (ii) is satisfied. However, condition (ii) does not require Δ_i to be countable, only that there is a countable set of beliefs that correspond to every preference parameter. This possibility can be used to approximate standard type spaces where $(*)$ does not hold by type spaces where it does hold. We illustrate this construction as follows.

Fix some $\delta_0 \in \Delta(V_{-i})$ and establish that with probability $\varepsilon > 0$, player i with parameter type $v_i \in V_i$ has belief δ_0 (instead of $\gamma(\cdot|v_i)$) and with probability $1-\varepsilon$, she has the original belief $\delta(\cdot) = \gamma(\cdot|v_i)$. The belief δ_0 could be thought of as an "ignorant" or default belief. In this case, $\Delta_i' = \{\delta_0\}$ would satisfy condition (ii) of Lemma 2.1 and, therefore, $(*)$ would hold.[14]

Another class of standard models where $(*)$ holds is given by the grid distributions defined on section 2.1.2. This class of models can also approximate (in a strong sense) any standard model, as we show in section 6.5. The approximation results discussed in this section and on section 6.5 have an important implication for applied work: $(*)$ cannot be refuted with any amount of finite data. Therefore, it does not add any significative extra restriction than that already imposed by the game model itself.

2.1.2 Very Simple (or Grid) Distributions

Perhaps the above discussion is yet too abstract for the taste of more applied readers, who are familiar with the standard approach and would like to see general, robust examples where our theory could be used. For this, we offer a concrete class of distributions where $(*)$ holds in the standard approach. This setting is our suggested approach for applied works. Section 6 illustrates the convenience of working with it.

For simplicity, assume that $V_i = [0, 1]$ for all $i \in I$. Grid distributions for multidimensional signals are formally defined in the appendix (see section 9.1). Also, assume that the common prior γ on $V = [0, 1]^N$ is defined by a density function $f : [0, 1]^N \to \mathbb{R}_+$. Consider the division of each $V_i = [0, 1]$ into k intervals of the form $\left(\frac{m-1}{k}, \frac{m}{k}\right]$, for $m = 1, ..., k$. We say that f defines a grid distribution if f is constant in each of the cubes thus formed. More precisely:

[14] The new type space thus formed would be $T_i^\varepsilon = T_i \cup (V_i \times \{\delta_0\})$. The definition of $\mathcal{T}_i^\varepsilon$ would be the same as before, just using the map $\widehat{V}_i^\varepsilon$ instead of $\widehat{V}_i$, where $\widehat{V}_i^\varepsilon : T_i^\varepsilon \to V_i$ is defined in the obvious way as a projection. The definition of τ_i^ε would take in account the ε probability above described.

Definition 2.2 *A function $f : [0,1]^N \to \mathbb{R}_+$ is a k-very simple density function or k-grid density function (and defines a k-very simple distribution or k-grid distribution), if for each $\boldsymbol{m} = (m_1, ..., m_N) \in \{1, ..., k\}^N$, f is constant on $\mathbb{I}_{\boldsymbol{m}}$, where*

$$\mathbb{I}_{\boldsymbol{m}} \equiv \left(\frac{m_1 - 1}{k}, \frac{m_1}{k}\right] \times ... \times \left(\frac{m_N - 1}{k}, \frac{m_N}{k}\right]. \tag{3}$$

The set of k-grid density functions is denoted by $\mathcal{D}^k$. We may say that f is just a grid density function k is not important in the context. The set of all grid density functions is denoted by $\mathcal{D}^\infty \equiv \cup_{k\in\mathbb{N}}\mathcal{D}^k$.

For instance, if $k = 3$ and we have $N = 2$ players, we can describe the density function f by a matrix, as shown in the picture below.

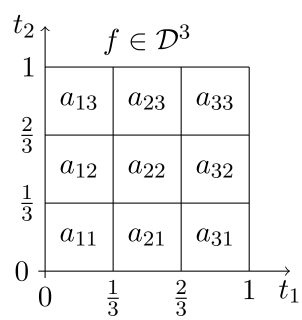

Figure 3: A density $f \in \mathcal{D}^k$ can be represented by a matrix $A = (a_{ij})_{k\times k}$.

It is easy to see that this class of distributions satisfies $(*)$. Indeed, if $v_i, v' \in \left(\frac{m-1}{k}, \frac{m}{k}\right]$, then

$$f(v_{-i}|v_i) = \frac{f(v_i, v_{-i})}{f(v_i)} = \frac{f(v_i', v_{-i})}{f(v_i')} = f(v_{-i}|v_i'), \quad \forall v_{-i} \in V_{-i},$$

because f assumes the the same values for v_i and v_i', no matter what v_{-i} is. In other words, any two signals in one of the interval share the same beliefs about the signals of the opponents. Since there is a finite number of intervals, there is a finite number of different beliefs. Therefore, condition (ii) of Lemma 2.1 is satisfied.

As discussed in section 6.5, we can approximate any continuous distribution in the standard approach with grid distributions, in a strong and useful sense.

2.1.3 Product structure

Now assume that each T_i has a product structure. By this, we mean that there is a homeomorphism m between T_i and (a subset of) $V_i \times \Delta(T_{-i})$. In this case, the space $(T_i, \mathcal{T}_i, \tau_i)$ is taken as primitive and the maps $\widehat{V}$: $T_i \to V_i$ and $\widehat{\delta}_i$ are defined by $m \circ p^{V_i}$ and $m \circ p^{\Delta_i}$, respectively, where $p^{V_i} : V_i \times \Delta(T_{-i}) \to V_i$ and $p^{\Delta_i} : V_i \times \Delta(T_{-i}) \to \Delta_i$ are the natural projection maps. This implies that τ_i defines measures $\gamma_i \equiv \tau_i \circ \widehat{V}_i^{-1}$ and $\nu_i \equiv \tau_i \circ \widehat{\delta}_i^{-1}$ on V_i and Δ_i, respectively. We slightly abuse terminology by saying that τ_i is absolutely continuous with respect to the product $\gamma_i \times \nu_i$ if: for any measurable set $E \subset V_i \times \Delta(T_{-i})$ such that $\gamma_i \times \nu_i(E) = 0$, then $\tau_i(m^{-1}(E)) = 0$.

This allows us to establish the following:

Lemma 2.3 *Assume that T_i has the product structure as described above and that condition (i) of Lemma 2.1 is satisfied. Moreover, assume that:*

(ii)' τ_i is absolutely continuous with respect to the product $\gamma_i \times \nu_i$.

Then, $\geqslant_i$ satisfies $(*)$.

All these results suggest that $(*)$ is a reasonable condition in models with correlated types.

3 Bayesian Games with a Monotonicity Condition

We now describe our general model of games of incomplete information. Each player $i \in I$ chooses actions in a set A_i, which satisfies the following:

Assumption 3.1 *For each i, (A_i, ρ_i) is a compact metric space and $\leqslant_i$ is a partial order on A_i.*

Notice that we do not assume that A_i is a lattice. For an example of a partial order that does not lead to a lattice; see footnote 29. Besides this, there are other examples of important partial orders used in economics that do not lead to lattices structures. For instance, Muller and Scarsini (2006) shows that some standard stochastic orders fail to be lattices.

The fact that we do not work with lattices, however, imposes some limitations. For instance, we cannot use supermodularity or quasi-supermodularity, because these properties are only defined for lattices. This leads us to make a contribution of defining a new monotonicity condition in section 3.1 below. But before introducing it, we need to complete the specification of the Bayesian game.

A_i is endowed with its Borel σ-algebra $\mathcal{A}_i$. The metric ρ_i and the binary relation $\leqslant_i$ are related by the following property:

Assumption 3.2 *For every $i \in I$ and every $a_i, a'_i, \underline{a}_i, \overline{a}_i \in A_i$, we have:*

$$\underline{a}_i \leqslant_i a_i, a'_i \leqslant_i \overline{a}_i \Rightarrow \rho_i(a_i, a'_i) \leqslant \rho_i(\underline{a}_i, \overline{a}_i). \tag{4}$$

The above assumption is trivially satisfied in euclidean spaces with the standard coordinatewise partial order. It is also satisfied if A_i is a space of real-valued functions and $\leqslant_i$ is the coordinatewise order, as long as the distance $\rho_i(a_i, a'_i)$ is obtained through the function $x \mapsto |a_i(x) - a'_i(x)|$, as it would be the case for the sup or the L^p-metrics. However, it may fail in some ordered spaces; for instance, it fails if $A_i = \mathbb{R}^2$ and $\leqslant_i$ is the lexicographic order.[15]

The product space $A \equiv \times_{i \in I} A_i$ is endowed with the sum metric ρ, that is, for $a = (a_i, a_{-i})$ and $a' = (a'_i, a'_{-i})$.

$$\rho(a, a') \equiv \sum_{i \in I} \rho_i(a_i, a'_i).$$

Given a profile of types $t = (t_1, ..., t_N)$ and a profile of actions $a = (a_1, ..., a_N)$ played by each individual, player i receives the payoff $u_i(t, a)$. We assume the following:

Assumption 3.3 *For each $i \in I$, the function $u_i : T \times A \to \mathbb{R}$ is bounded and measurable.*

Let $\mathcal{F}_i$ denote the set of measurable functions from T_i to A_i. The strategy adopted by player i will be a function $s_i \in \mathcal{F}_i$. Let $\mathcal{F}_{-i}$ denote $\times_{j \neq i} \mathcal{F}_j$. Given a profile of strategies $s = (s_1, ..., s_N)$, player i has (ex ante) utility[16]

$$U_i(s_i, s_{-i}) \equiv \int u_i(t, s(t))\, d\tau = \int u_i(t, s(t)) f(t)\, \tau(dt), \tag{5}$$

[15] Indeed, we could have $\underline{a}_i = (1, 0) \leqslant_i a_i = (1, 1) \leqslant_i a'_i = (1, 2) \leqslant_i \overline{a}_i = (1.1, 0)$ and $\rho_i(a_i, a'_i) = 1 > 0.1 = \rho_i(\underline{a}_i, \overline{a}_i)$.

[16] In this version, we will restrict our definitions to the case with a common prior; these definitions can be easily extended to cases without common prior. See for instance, van Zandt (2010).

where we assume that $t \mapsto u_i(t, s(t))$ is measurable, hence integrable.

As usual, a profile $s = (s_1, ..., s_N)$ is a (Bayesian pure strategy) equilibrium if $U_i(s_i, s_{-i}) \geqslant U_i(s_i', s_{-i})$ for all i and $s_i' \in \mathcal{F}_i$.

It will be useful to introduce the interim payoff. Player i's interim payoff when she is of type t_i and plays action a_i is given by:

$$\begin{aligned} \Pi_i(t_i, a_i, s_{-i}) &\equiv \int_{T_{-i}} u_i(t_i, t_{-i}, a_i, s_{-i}(t_{-i}))\ \tau(dt_{-i}|t_i) \\ &= \int_{T_{-i}} u_i(t_i, t_{-i}, a_i, s_{-i}(t_{-i}))\ f(t_{-i}|t_i)\ \prod_{j \neq i} \tau_j(dt_j). \end{aligned} \tag{6}$$

The above setup will be valid throughout the paper, although we specialize later to some particular cases. Our main interest is on the distribution of types, which is characterized by f. Therefore, when the other elements of the game are clear from the context and the game has a *pure strategy* equilibrium, we will sometimes abuse terminology and say that f has an equilibrium.

Although we will focus primarily on pure strategies, some of our results consider mixed strategies. For this, we will define mixed (behavioral) strategies.[17] A behavioral strategy for player i is a Markov kernel $\mu_i : T_i \times \mathcal{A}_i \to [0, 1]$. Let S_i denote the set of behavioral strategies by player i. Following Balder (1988), we define product $\boldsymbol{\mu} = \mu_1 \otimes ... \otimes \mu_m$ by:

$$\boldsymbol{\mu}\left((t_1, ..., t_N), B_1 \times ... \times B_N\right) \equiv \prod_{i \in I} \mu_i(t_i, B_i)$$

for rectangles and extend it for measurable sets by standard arguments. (See details in Balder (1988).) For each distributional strategy σ_i corresponds a behavioral strategy μ_i. Therefore, the ex ante utility function can be given by:

$$U_i(\mu_1, ..., \mu_N) \equiv \int_T \left[\int_A u_i(t, a) \boldsymbol{\mu}(t, da)\right] \tau(dt). \tag{7}$$

Recall that a behavioral strategy μ_i is *pure* if $\mu_i(t_i, \cdot)$ is a Dirac measure (has just one point in its support) for almost all t_i. Player i's interim payoff when she is of type t_i and plays action a_i is given by the function Π_i :

[17] Behavioral and distributional strategies are equivalent. See, for instance, Rustichini (1993).

$T_i \times A_i \times S_{-i} \to \mathbb{R}$ defined by:

$$\begin{aligned}\Pi_i(t_i, a_i, \mu_{-i}) &\equiv \int_{T_{-i}} P_i(t_i, t_{-i}, a_i, \boldsymbol{\mu}_{-i})\ \tau(dt_{-i}|t_i) \\ &= \int_{T_{-i}} P_i(t_i, t_{-i}, a_i, \boldsymbol{\mu}_{-i})\ f(t_{-i}|t_i)\ \prod_{j \neq i} \tau_i(dt_j).\end{aligned}$$

where $P_i : T \times A_i \times S_{-i} \to \mathbb{R}$ is defined by:

$$P_i(t, a_i, \boldsymbol{\mu}_{-i}) \equiv \int_{A_{-i}} u_i(t, a_i, a_{-i})\boldsymbol{\mu}_{-i}(t_{-i}, da_{-i}).$$

Occasionally, we will consider the restriction $\Pi_i : T_i' \times A_i \times S_{-i} \to \mathbb{R}$ of Π_i to subsets $T_i' \subset T_i$, which is defined in the standard way.

3.1 A new monotonicity property

We introduce a new monotonicity property, that generalizes supermodularity and increasing differences or quasi-supermodularity and single-crossing. For this, let us introduce some notation. Let $(X, \geqslant)$ and $(Y, \geqslant)$ be partially ordered sets. Also, let Z be an index set and consider the function $g : X \times Y \times Z \to \mathbb{R}$.[18]

Notice that we did not require the above sets to be lattices.[19] Since we work only with partially ordered sets and not lattices, we cannot even define supermodularity or quasi-supermodularity. Our monotonicity property is presented in two forms: increasing differences (ID) and single-crossing (SC). In the ID form, it is a suitable generalization of the combination of supermodularity and increasing differences, which does not require the lattice structure:

Definition 3.1 (ID-Monotonicity Property) *g satisfies the ID-monotonicity property in $X \times Y$ if for any $x, x' \in X$, such that $x < x'$, and $y, y' \in Y$ such that $\neg(y' \geqslant y)$,[20] there exists $\underline{y}, \overline{y} \in Y$ satisfying the following:*

$$g(x, y, z) - g(x, \underline{y}, z) < g(x', \overline{y}, z) - g(x', y', z), \forall z \in Z. \tag{8}$$

[18] In our applications below, g will be the the ex post or the interim payoff function, X will be a subset of T_i, Y will be A_i and Z will be S_{-i} for each i.

[19] The relevance of this relaxation is discussed after Assumption 3.1.

[20] As usual, the symbol $\neg$ means negation.

It is important to highlight that $\underline{y}$ and $\overline{y}$ do not need to be ordered with respect to y and y'. In contrast, if we assume that Y is a lattice and supermodularity is valid, these elements would be $\underline{y} = y \wedge y'$ and $\overline{y} = y \vee y'$ and would, therefore, satisfy $\overline{y} \geqslant y, y' \geqslant \underline{y}$.

Also, note that g satisfies the above assumption if it is supermodular and has increasing differences, as the following lemma establishes:

Proposition 3.2 *Assume that Y is a lattice and that $g : X \times Y \times Z \to \mathbb{R}$ is supermodular in Y and satisfies increasing differences in $X \times Y$ or, alternatively, it is strictly supermodular in Y and satisfies non-decreasing differences in $X \times Y$. Then g satisfies the ID-monotonicity property in $X \times Y$.*[21]

An interesting characteristic of the ID-monotonicity property is that it is preserved under integration—see Lemma 8.5 in the appendix for a precise statement. Therefore, if the ex post utility function satisfies it, the interim payoff function also does. This is useful because it is in general much easier to check a condition on the ex post payoff function. If we are willing to make assumptions directly on the interim function, then the increasing difference form of the monotonicity property can be relaxed to a single-cross form:

Definition 3.3 (SC-Monotonicity Property) *g satisfies the SC-monotonicity property in $X \times Y$ if for any $x, x' \in X$, such that $x < x'$, and $y, y' \in Y$ such that $\neg(y' \geqslant y)$, there exists $\underline{y}, \overline{y} \in Y$ satisfying the following:*

$$g(x, y, z) \geqslant g(x, \underline{y}, z) \implies g(x', \overline{y}, z) > g(x', y', z), \forall z \in Z. \tag{9}$$

It is clear that ID-monotonicity implies SC-monotonicity. Also, Proposition 3.2 has a version for the SC-monotonicity property:

Proposition 3.4 *Assume that Y is a lattice and that $g : X \times Y \times Z \to \mathbb{R}$ is weak quasi-supermodular in Y and satisfies strict single crossing in $X \times Y$ or, alternatively, it is strictly quasi-supermodular in Y and satisfies single crossing in $X \times Y$. Then g satisfies the SC-monotonicity property in $X \times Y$.*

Notice that both ID-monotonicity and SC-monotonicity require strictness in the inequality. Strict conditions have received less attention than their weak counterpart. As some of the techniques of this paper will show, small perturbations of standard games will satisfy this strict form.

[21] See section 8.1.2 in the appendix for definitions of the properties used in Propositions 3.2 and 3.4, and for proofs of both propositions.

Actually, if we have freedom to choose the order on the type spaces, then the ID or SC-monotonicity properties are almost trivial. Let us illustrate this claim for interim payoff functions in a simple game with just two actions $Y = \{y, y'\}$, with $y > y'$. For this, let us fix the strategies μ_{-i} of the other players. Then, define that types x and x' are ordered ($x < x'$) if and only if:

$$\Pi(x, y) - \Pi(x, y') < \Pi(x', y) - \Pi(x', y'). \tag{10}$$

This makes the ID-monotonicity property trivially satisfied.

Of course the point of the above trick is not to provide a way to define orders, but to illustrate how the monotonicity property is, in itself, a weak condition. When the order used is such that $(*)$ is also valid, the monotonicity property can provide interesting results, as we will illustrate below.

4 Basic Results

We now present the results for the general framework described above.

4.1 Monotonic Best Replies, without $(*)$

We present two versions of our basic framework, which does not assume $(*)$. The first one contains assumptions on the ex post utility function, which is in general easier to verify. For each $\delta \in \Delta_i \equiv \widehat{\delta}_i(T_i)$, define $T_{i\delta} \equiv \{t_i \in T_i : \widehat{\delta}_i(t_i) = \delta\}$. Let S_{-i} denote the set of profiles of mixed strategies played by players $j \neq i$. Given $\mu_{-i} \in S_{-i}$ and $t_i \in T_i$, let $BR(t_i, \mu_{-i}) \subset A_i$ denote the set of best replies by player i of type t_i to the mixed strategies μ_{-i}.

Theorem 4.1 *Let Assumptions 3.1, 3.2 and 3.3 hold. Assume that for each $\delta \in \Delta_i$, $u_i : T_{i\delta} \times T_{-i} \times A \to \mathbb{R}$ satisfies the ID-monotoniticy property in $T_{i\delta} \times A_i$. Let $\mu_{-i} \in S_{-i}$. The following holds:*

1. *If $t_i, t_i' \in T_{i\delta}$, $t_i < t_i'$, $a_i \in BR(t_i, \mu_{-i})$, $a_i' \in BR(t_i', \mu_{-i})$ then $a_i \leqslant a_i'$.*
2. *If μ_i be a better reply strategy to μ_{-i}, then for each $\delta \in \Delta_i$, the set of player i's types in $T_{i\delta}$ who possibly play mixed strategies under μ_i is a denumerable union of antichains.*[22,23]

[22]By denumerable we mean either finite or countable.

[23]An antichain in a partially ordered set $(X, \leqslant)$ is a subset of X that contains no two ordered points.

Note that the above result imposes no assumption on the type spaces: they can be completely general (just measurable spaces). This result can be stated with (slightly more general) assumptions in the interim payoff function.

Theorem 4.2 *Let Assumptions 3.1, 3.2 and 3.3 hold. Assume that for each* $\delta \in \Delta_i$, $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ *satisfies the SC-monotonicity property in* $T_{i\delta} \times A_i$. *Let* $\mu_{-i} \in S_{-i}$. *Then the two conditions of Theorem 4.1 hold.*

Both Theorems 4.1 and 4.2 apply to any type space. For instance, the set of types can be *finite*, continuous or even a universal type space without common priors. Note that $(*)$ *was not* assumed in either theorem. We also remark that, although the first claim above is easier to understand and appreciate, the second claim is the most useful for the rest of this paper.

It is useful to observe that the beliefs do not play a specific role in Theorem 4.2, because it is stated on the interim payoff function. That is, the proof of Theorem 4.2 actually establishes the following:

Theorem 4.2' *Let Assumptions 3.1, 3.2 and 3.3 hold. Let* $\{T_{i\lambda}\}_{\lambda\in\Lambda}$ *be a partition of* T_i *and assume that for each* $\lambda \in \Lambda$, $\Pi_i : T_{i\lambda} \times A_i \times S_{-i} \to \mathbb{R}$ *satisfies the SC-monotoniticy property in* $T_{i\lambda} \times A_i$. *Let* $\mu_{-i} \in S_{-i}$. *If* $t_i, t_i' \in T_{i\lambda}$ *for some* $\lambda \in \Lambda$, $a_i \in BR(t_i, \mu_{-i})$, $a_i' \in BR(t_i', \mu_{-i})$ *then* $a_i \leqslant a_i'$.[24]

The proof of these results can be summarized as follows. First, we reduce Theorem 4.1 to Theorem 4.2. This step is accomplished in Lemma 8.5, by observing that the ID-monotonicity property in the ex post utility function is preserved under integration and if the considered types share the same belief. Therefore, the SC-monotonicity property is satisfied by the interim payoff function in the partition defined by the beliefs of types.

The proof of Theorem 4.2 can be divided in two parts. The first and easiest part is to establish the monotonicity of best actions. This comes from the monotonicity property in a straightforward way, by obtaining a contradiction if the best reply actions are not ordered—see Lemma 8.7. The argument for the second claim in the theorem is more involved. First, we need to establish a mathematical result about chains.[25] Namely, if a chain has at least three elements, then we can divide it in three sets such that for every pair of points in one of the sets, we can find a third point in the other two sets which are strictly between these points.[26] The proof of this

[24] We omit the straightforward adaptation of Theorem 4.1's implication (2) for this setting.
[25] A chain is a totally ordered subset of a partially ordered set.
[26] If the chain is finite, we can divide it in just two sets with the mentioned property.

fact requires Zorn's Lemma and some non-trivial constructions. This fact is used in conjunction with Assumptions 3.1 and 3.2 to show that any chain of types that have two best reply actions at least $\frac{1}{n}$ apart must be a definitive finite number. Then, we use a result in the theory of partially ordered sets (Dilworth's Theorem, see Lemma 8.10) to argue that this implies that the number of antichains of types with two best reply actions at least $\frac{1}{n}$ is finite. This gives the conclusion stated in (2).

4.2 Pure strategy equilibria with $(*)$

Now, we will introduce our main assumption $(*)$ on the type space and show that all best replies are in pure strategies.

Theorem 4.3 *Let $(*)$ and the assumptions of Theorem 4.1 or 4.2 hold. Let μ_{-i} be any mixed strategy played by player i's opponents and let μ_i be i's best reply to μ_{-i}. Then, μ_i is pure.*

Therefore, under these assumptions, if there is a mixed strategy equilibrium, it is a pure strategy equilibrium. Moreover, all equilibria are in pure strategies.

Theorem 4.3 gives conditions under which all equilibria must be in pure strategies; more than that, they show that every best-reply, even to mixed strategies, is pure. This is a stronger conclusion than most results about equilibrium existence, which consider only best-reply to monotonic strategies.

The two closest results available in the literature are Maskin and Riley (2000, Proposition 1) and Araujo and de Castro (2009, Theorem 1). There are two main differences between these results and Theorem 4.3: first, both papers restrict attention to unidimensional auction games and second, both assume independence of types. Therefore, even restricted to unidimensional affiliated types, Theorem 4.3 presents a new result.

In the remainder of the paper, we will show how Theorem 4.3 leads to new equilibrium existence results in games of incomplete information. But before discussing these results, we would like to comment on how $(*)$ and Theorem 4.3 could be extended to a setting without measure τ_i.

4.3 Insignificant sets

The use of a measure τ_i on the statement of $(*)$ may be considered undesirable, given that in principle the type spaces T_i could be considered only measurable spaces. In this section, we show that $(*)$ could be rephrased without mentioning a measure τ_i, using instead a notion of "insignificant sets."

Definition 4.4 *A collection $\mathcal{I}_i$ of subsets of T_i is a collection of insignificant sets if: (i) $\mathcal{I}_i \subset \mathcal{T}_i$; (ii) $\mathcal{I}_i$ is closed for countable unions. Any set in $\mathcal{T}_i \setminus \mathcal{I}_i$ is called a significant set.*

For the discussion below, fix a collection of insignificant sets $\mathcal{I}_i$. $(*)$ can be rephrased as follows:

> **Assumption $\widetilde{(*)}$:** If $E \in \mathcal{T}_i$ is a significant set, then there exist $t_i, t_i' \in E$ such that $t_i <_i t_i'$ and $\widehat{\delta}_i(t_i) = \widehat{\delta}_i(t_i')$.

The proof of Theorem 4.3 allows us to conclude the following:

Proposition 4.5 *Let $\widetilde{(*)}$ and the assumptions of Theorem 4.1 or 4.2 hold. Let μ_{-i} be any mixed strategy played by player i's opponents and let μ_i be i's best reply to μ_{-i}. Then, the set of types $t_i \in T_i$ which play mixed strategies according to μ_i is an insignificant set.*

5 PSE existence in Private Value Auctions

In this section, we describe the first application of our theory: a general pure strategy equilibrium result for correlated types in private value auctions. We begin by describing the type structure and then describe the model for the game, that follows very closely that of Jackson and Swinkels (2005), henceforth JS.

5.1 Multi-unit Private Value Auctions

The description of the model is organized in two subsections: values and types (5.1.1) and the game itself (5.1.2).

5.1.1 Multidimensional values and belief types

A useful instance of our main framework is now described. Assume that the set of player i's types is $T_i \subset E_i \times \tilde{V}_i \times \Delta_i$, where:[27]

- $E_i \equiv \{0, 1, \ldots, \ell\}$ is the set of player i's possible initial endowments;

[27] Recall that T_i is the support of τ_i.

- $\tilde{V}_i = [\underline{v}_i, \overline{v}_i]^\ell \subset \mathbb{R}^\ell$ denotes the vector of player i's valuations for objects in a multi-unit auction;[28]

- Δ_i is the set of i's beliefs about other players' types.

A typical type is therefore $t_i = (e_i, v_i, \delta_i)$, where:

- $e_i \in \{0, 1, \ldots, \ell\}$ denotes the number of units that player i is endowed with;

- $v_i = (v_{i1}, \ldots, v_{i\ell})$ is the vector of player i's valuations, meaning that i has marginal value v_{ih} for the hth object, satisfying nonincreasing marginal valuations, that is, $v_{ih} \geqslant v_{i,h+1}$ for all i, h; and

- $\delta_i \in \Delta_i$ denotes i's signal about other bidders' valuations.

The values $\underline{v}_i$ and $\overline{v}_i$ are finite for all i. The objects are indivisible and identical. The vector of endowments is $e = (e_0, ..., e_N) \in E \equiv \{0, 1, ..., \ell\}^{N+1}$; the vector of values is $v = (v_1, ..., v_N) \in \tilde{V} \equiv \prod_{i=1}^N [\underline{v}_i, \overline{v}_i]^\ell$. Let $\Delta \equiv \times_{i=0}^N \Delta_i$, so that $T = E \times \tilde{V} \times \Delta$ and $t = (e, v, \delta) \in T$ denotes a profile of types.

Observe that given that δ_i determines i's beliefs about other players' types, then for $t_i = (e_i, v_i, \delta_i)$ and $t_i' = (e_i', v_i', \delta_i)$, we must have:

$$\Pr\left[t_{-i} \in E | t_i\right] = \Pr\left[t_{-i} \in E | \delta_i\right] = \Pr\left[t_{-i} \in E | t_i'\right].$$

We also define an order in $V_i = E_i \times \tilde{V}_i$. Actually, we define the order exclusively on $\tilde{V}_i = [\underline{v}_i, \overline{v}_i]^\ell$ as follows. First, define $v_i <_i v_i'$ by:

$$v_i <_i v_i' \iff v_{ih} < v_{ih}', \forall h = 1, ..., \ell. \tag{11}$$

Then, define $v_i \leqslant_i v_i'$ iff $v_i <_i v_i'$ or $v_i = v_i'$. Notice that this is a partial order, but it does not generate a lattice.[29] On the other hand, it is not difficult to see that this order satisfies condition (i) of Lemma 2.1.

This order on $\tilde{V}_i$ defines an order on T_i by (1). More concretely, for $t_i = (e_i, v_i, \delta_i)$ and $t_i' = (e_i', v_i', \delta_i')$, we define

$$t_i <_i t_i' \iff v_i < v_i' \text{ and } \delta_i = \delta_i', \tag{12}$$

[28] Observe that $V_i \equiv E_i \times \tilde{V}_i$ corresponds to the "preference" part of the type.

[29] A partial order set $(X, \leqslant)$ generates (or is) a lattice if for any $x, y \in X$ there is lowest upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ for any pair $\{x, y\}$. To see that the order above is not a lattice, it is enough to consider $\ell = 2$. Consider the points $x = (1, 0)$ and $y = (0, 1)$. Then $x, y \leqslant z^\varepsilon \equiv (1+\varepsilon, 1+\varepsilon)$ for any $\varepsilon > 0$ but $\neg(x \leqslant z^0)$ and $\neg(y \leqslant z^0)$. Therefore, there is no lowest upper bound for $\{x, y\}$.

and $t_i \leqslant_i t_i'$ iff $t_i <_i t_i'$ or $t_i = t_i'$. Note that (12) is a variation of definition (1) given in section 2.1.

Although we will use this order below, we should emphasize that this is not the only case where our techniques apply. Actually, any order that is stronger than (12) could be automatically used. For instance, in games with risk aversion, one could consider the order defined by Reny (2011).

Additionally, we assume either condition (ii) of Lemma 2.1 or condition (ii)' of Lemma 2.3, that is, that τ_i is absolutely continuous with respect to the product $\gamma_i \times \nu_i$, where γ_i be the marginal of τ_i over $\tilde{V}_i$ and ν_i, the marginal over Δ_i. As discussed in section 2.1, these conditions are sufficient for implying $(*)$. $(*)$ is essentially the *only* extra assumption that we require with respect to JS.[30]

The assumptions and description above are maintained for our results in this section, even without explicit reference.

5.1.2 Game description

We will now describe the general private value auction model introduced by JS. It will be useful to explicitly consider the non-strategy player 0. Each bidder places a bid $b_i \in A_i \subseteq [\underline{b}, \overline{b}]^\ell$. The order on the action space is the standard coordinate-wise order on $\mathbb{R}^\ell$. Let $A = A_0 \times ... \times A_N$. The set of allocations is $\Omega \equiv \{0, 1, ..., \ell\}^{N+1}$. Given types $t \in T \subset E \times \tilde{V} \times \Delta$, bids $b \in A$ and allocation $\omega = (h_0, ..., h_N) \in \Omega$, player i's ex post utility is:[31,32]

$$u_i(t, b, \omega) \equiv \sum_{j=0}^{h_i} v_{ij} - p_i(h_i, e, b), \tag{13}$$

where $p_i : \{0, ..., \ell\} \times \Omega \times A \to \mathbb{R}$ is player i's payment function, which can depend not only on how many objects she gets, but also on everybody's bids and endowments. We will introduce restrictions on the payment functions p_i below.

[30] We do require an extra technical assumption—see assumption 5.1 below. However, this is done more for simplicity.

[31] Define $v_{i0} = 0$ for all i.

[32] For simplicity, we do not consider risk aversion, although our techniques could be extend to this setting as well.

The outcome correspondence $O : \Omega \times A \to \Omega$ is defined by:

$$O(e,b) \quad \equiv \{m \in \Omega : \quad \sum_{i=0}^{N} m_i = \sum_{i=0}^{N} e_i, \text{ and}$$
$$\big(b_{jh'} > b_{ih} \text{ and } m_i \geqslant h\big) \implies m_j \geqslant h'\}. \qquad (14)$$

The first condition above is just the requirement that objects are not created nor destroyed. The second condition amounts to requiring that higher bids are given priority over lower bids in allocating objects. A tie-breaking rule is just a selection of this correspondence.[33] More formally, a tie-breaking rule will be denoted by a profile of functions $h^* = (h_i^*)_{i=0}^N$ such that $h^*(e,b) \in O(e,b)$. Note that the only points were there are some freedom for the value of h^* ($O(e,b)$ is not a singleton) occur when there is some "relevant tie," that is, a tie that occurs exactly at the number of units $\sum_{i=0}^N e_i$ available for negotiation.

When there is a tie, (14) still determines maximum and minimum values for the number of units $h_i^*(e,b)$ that player i can receive. We will denote these values by $\underline{h}_i(e,b)$ and $\overline{h}_i(e,b)$. For simplicity of notation, we will constantly omit e in the argument of the functions h_i^*, $\underline{h}_i$ and $\overline{h}_i$ below. No confusion should arise.

We assume the following for the payment rule:

Assumption 5.1 *The payment function is given by:*

$$p_i(h,e,b_i,b_{-i}) = \sum_{j=1}^{h} p_{ij}(h,b_{ij}) + q_i(h,b_{-i}) + \underline{r}_i(h,\underline{p}) + \overline{r}_i(h,\overline{p}), \qquad (15)$$

where $\underline{p}$ denotes the highest losing bid and $\overline{p}$ denotes the lowest winning bid, if there is no competitive tie; if there is a competitive tie at β, $\underline{p} = \overline{p} = \beta$. Moreover, all these functions are nondecreasing.

The first term in the sum above corresponds to "pay-your-bid" elements and will not be zero in discriminatory auctions. The second term, $q_i(h,b_{-i})$ depends exclusively (and arbitrarily) on the bids of other players. This allows us to cover Vickrey auctions, for instance. The other two terms, $\underline{r}_i(h,\underline{p})$ and $\overline{r}_i(h,\overline{p})$, allow to capture the two distinct formats of uniform

[33] The formal definition given by JS is not exactly this, as they allow for randomizations on the allocations. They also allow for "omniscient" tie-breaking rules, that is, rules that may vary depending on the values. However, they prove that the actual tie-breaking rule is not important in their setting. Therefore, this formulation is enough for our purposes.

price auctions: the ones where the clearing price is the highest losing bid $\underline{p}$ or the lowest winning bid $\overline{p}$. Note that these values are not determined exclusively by b_{-i} and, therefore, cannot be captured only on $q_i(h, b_{-i})$. Note also that all terms can vary with the total number of allocated objects h. Therefore, variations and combinations of the payment rules of standard auctions are allowed.

5.2 Pure Strategy Equilibrium Existence

Consider the framework described in subsections 5.1.1 and 5.1.2, which is essentially the same as JS', with their assumptions 1-9. We have the following:

Theorem 5.1 *Let assumptions $(*)$ and 5.1 hold, together with JS' assumption 10.*[34] *Then, there exists a monotonic pure strategy equilibrium in undominated* strategies with a zero probability of competitive ties, which is an equilibrium under any omniscient and effectively trade-maximizing tie-breaking rule, including the standard tie-breaking rule.*

Note that the main difference from Theorem 5.1 above and JS' Corollary 14 is that they assume that the distribution of types is independent.

The proof of Theorem 5.1 goes as follows. We first establish that the auction is modular, using Proposition 5.4 and Assumption 5.1.

The first difficulty in proving Theorem 5.1 comes from the fact that auctions do not satisfy in general increasing differences in $T_i \times A_i$, although they satisfy nondecreasing differences. Therefore, our ID or SC-monotonicity properties may fail to hold. To circumvent this problem, we consider modified auctions (n-auctions), which are auctions where with a probability $\frac{1}{n}$, a non-strategic player bids uniformly on $[\underline{b}, \overline{b}]$. Actually, the n-modified auction includes also a modification of the tie-breaking rule. Since the discussion about the tie-breaking requires modification requires many more details, we discuss this issue in section 5.3 below.

We show that our ID-monotonicity property holds in each of the n-modified auctions. This allows us to conclude that each n-auction has an equilibrium in pure strategies which are monotonic when restricted to the set of types sharing the same beliefs.

When $n \to \infty$, there is a pointwise convergent subsequence of strategies because the set of strategies is compact. Pointwise convergence leads

[34] JS' assumption 10 specifies a technical measurability condition to ensure the existence of equilibrium in undominated* strategies. For a definition of this class of strategies and an explicity statement and discussion of JS' assumption 10, we refer the reader to that paper.

(through Lebesgue theorem) to convergence of the (interim) payoffs, both for strategies in the optimum, as deviating strategies. In fact, the argument needs to be more careful here, because of the possibility of ties with positive probability. For this, we need to define a special tie-breaking rule that guarantees convergence of the payoffs. After this, we are able to argue that the tie-breaking rule does not matter. Finally, we show that if there is a deviating strategy that does better, it would also do better along the sequence, which is a contradiction. The appendix details this argument.

Remark 5.2 . *Theorem 5.1 states the existence of a monotonic pure strategy equilibria. This is possible because of the order (12) considered. If we use instead (1), then the equilibrium could fail to be monotonic. This is just to highlight that there is a subtlety when we talk about "monotonicity" in multi-dimensional spaces. Since we may have more than one "reasonable" order in some spaces, we may have different notions of monotonicity.*[35] *For instance, Reny (2011) proposes a different order for cases with risk aversion and argues that it is a natural order. When we do have a clear unique candidate for the types' order (for instance, in unidimensional settings), then the monotonicity question is important. For this reason, section 6 address the existence of monotonic equilibrium in first price auctions.*

Remark 5.3 *Theorem 5.1 is stated only for private values auctions, as in the JS setting. However, the existence of pure strategy equilibrium can be extended to interdependent values if we are willing to allow special kinds of tie-breaking rules, as Araujo and de Castro (2009) did. In other words, Theorem 5.1 is stated only to private values to prevent us from dealing with special tie-breaking rules.*

5.3 Tie-breaking rule for the modified auction

To establish our ID-monotonicity property, we will need an assumption on the tie-breaking rule. This assumption generalizes a rule that was introducec by McAdams (2003) and used by him and by Reny (2011) to establish equilibrium existence for the uniform price auction. McAdams (2003, p.1198) describes his rule as follows:[36]

> Each bidder is assigned at least $\underline{h}_i(b)$ and randomly ordered into a ranking ρ to ration the remaining quantity $r \equiv K - \sum_{i=1}^{N} \underline{h}_i(b)$.

[35] Actually, if we have complete freedom in defining the types' order, any pure strategy equilibrium $a = (a_i, a_{-i})$ can be monotonic: just define $t_i \leqslant t'_i$ if and only if $a_i(t_i) \leqslant a_i(t'_i)$.

[36] We adapted his notation to ours.

If $r = 0$, stop. Else the first bidder in order, $i_1 = \rho(1)$, receives $q^*_{i_1} = \underline{h}_i(b) + \min\{\overline{h}_i(b) - \underline{h}_i(b), r\}$. Decrement r by $\overline{h}_i(b) - \underline{h}_i(b)$ and repeat this process with bidder $i_2 = \rho(2)$ and so on until all quantity has been assigned.

It is not difficult to see that this rule satisfies the following:

Assumption 5.2 *Let $\tilde{b}_i$, $\hat{b}_i \in A_i$ and fix $b_{-i} \in A_{-i}$ and $h \in \{1, ..., \ell\}$. The following holds:*

1. *If $\tilde{b}_i \leqslant_i \hat{b}_i$ then $h^*_i(\tilde{b}_i, b_{-i}) \leqslant h^*_i(\hat{b}_i, b_{-i})$;*
2. *If $h^*_i(\tilde{b}_i, b_{-i}) \geqslant h - 1$, $h^*_i(\hat{b}_i, b_{-i}) \geqslant h - 1$ and $\tilde{b}_{ih} = \hat{b}_{ih} = s_{ih}$, then:*

$$h^*_i(\tilde{b}_i, b_{-i}) \geqslant h \iff h^*_i(\hat{b}_i, b_{-i}) \geqslant h. \tag{16}$$

The first requirement in Assumption 5.2 is just a mild monotonicity condition: no bidder can receive *more* units by bidding less. Most tie-breaking rules satisfy this condition. The second condition is a little bit more restrictive. It requires that a bidder wins unit h-th irrespective of what are his bids for the $(h + 1)$-th unit and above, and also it depends on her bids for units below h only through the fact of winning or not winning those units. It is easy to see that McAdams' rule satisfies Assumption 5.2 . Another rule that satisfies Assumption 5.2 is the following. Let bidders be ordered in some arbitrary fashion. If there is a tie, give one object to the first bidder in the tie, according to this order; then give the second unit to the second one in the order, and so on. If all bidders in the tie receive one object, but there are still unassigned objects, repeat the process, until no object is left unassigned. Since the allocation of the h-th unit does not depend on the bids after the h-th, it is easy to see that this rule also satisfies Assumption 5.2 . On the other hand, consider the following rule: in the case of a tie, divide the number of objects in the tie by the tying bidders and give to each of those bidders the integer part of this division.[37] The remaining objects are randomly allocated. This rule does not satisfy Assumption 5.2 and also fails to have the property described in the following:[38]

[37] This is similar to the pro-rata on the margin rule that is used in some real auctions.

[38] For an example of the failure of Proposition 5.4, assume that we have two bidders, four units, bidder 2 bids 2 for all units and $b^1_i = (4, 3, 1, 1)$, $b^2_i = (2, 2, 2, 2)$. Then, $b^1_i \wedge b^2_i = (2, 2, 1, 1)$, $b^1_i \vee b^2_i = (4, 3, 2, 2)$, $h^*_i(b^1_i, b_{-i}) = h^*_i(b^2_i, b_{-i}) = 2$, but $h^*_i(b^1_i \wedge b^2_i, b_{-i}) = 1$ and $h^*_i(b^1_i \vee b^2_i, b_{-i}) = 3$.

Proposition 5.4 *Let Assumption 5.2 hold. Let $b_i^1,\ b_i^2 \in A_i$ and assume (w.l.o.g.) that $h_i^*(b_i^1, b_{-i}) \leqslant h_i^*(b_i^2, b_{-i})$. Then,*[39]

$$h_i^*(b_i^1, b_{-i}) = h_i^*(b_i^{1\wedge 2}, b_{-i}) \text{ and } h_i^*(b_i^2, b_{-i}) = h_i^*(b_i^{1\vee 2}, b_{-i}).$$

Proposition 5.4's proof is simpler and more direct than McAdams' argument.

As a step in the proof of Theorem 5.1, we show in the appendix (see Lemma 8.11 and Corollary 8.13) that Assumption 5.2 and (15) lead to the following property for any i, b_i^1, b_i^2 and b_{-i}:

$$p_i(h, e, b_i^1, b_{-i}) - p_i(h, e, b_i^{1\wedge 2}, b_{-i}) = p_i(h, e, b_i^{1\vee 2}, b_{-i}) - p_i(h, e, b_i^2, b_{-i}). \quad (17)$$

It should be noted that this result is more complex than in McAdams (2003) and Reny (2011), because they focused exclusively on the uniform price auction, with its simpler payment rule.

6 Existence of SMPSE

Given the result presented in the last section (Theorem 5.1), it is natural to ask whether (and when) it is possible to establish existence of equilibrium in *monotonic* pure strategies. In this section we give a complete answer to this question in a more particular setting, that of very simple or grid distributions, defined in section 2.1.2.

To consider the relevant issues, first recall the standard result of auction theory on SMPSE in private value auctions: if there is a differentiable symmetric increasing equilibrium, it satisfies the differential equation (see Krishna 2002 or Menezes and Monteiro 2005):

$$b'(v) = \frac{v - b(v)}{F(v|v)} f(v|v).$$

If f is Lipschitz continuous, one can show that this equation has a unique solution. Under some assumptions (affiliation or, a little bit more generally, Property VI' in de Castro (2011)), it is possible to ensure that this solution is, in fact, equilibrium. Now, if the distribution is very simple ($f \in \mathcal{D}^\infty$), the right hand side of the above equation is not continuous, and one cannot directly apply standard techniques. We proceed as follows.

First, we show that if there is a symmetric increasing equilibrium b, under mild conditions (satisfied by $f \in \mathcal{D}^\infty$), b is continuous. We also

[39] Hereafter, $b_i^{1\wedge 2}$denotes $b_i^1 \wedge b_i^2$ and $b_i^{1\vee 2}$ denotes $b_i^1 \vee b_i^2$.

prove that b is differentiable at the points where f is continuous. Thus, for $f \in \mathcal{D}^{\infty}$, b is continuous everywhere and differentiable everywhere but, possibly, at the points of the form $\frac{m}{k}$. See Figure 4.

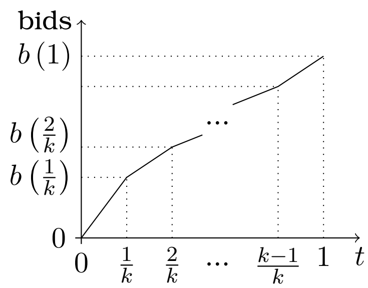

Figure 4: Bidding function for $f \in \mathcal{D}^k$.

With the initial condition $b(0) = 0$ and the above differential equation being valid for the first interval $\left(0, \frac{1}{k}\right)$, we have uniqueness of the solution on this interval and, thus, a unique value of $b\left(\frac{1}{k}\right)$. Since b is continuous, this value is the initial condition for the interval $\left(\frac{1}{k}, \frac{2}{k}\right)$, where we again obtain a unique solution and the uniqueness of the value $b\left(\frac{2}{k}\right)$. Proceeding in this way, we find that there is a unique b which can be a symmetric increasing equilibrium for an auction with $f \in \mathcal{D}^{\infty}$.

To formalize this result, assume that we have a first price auction with n symmetric players, such that if player i with signal $v_i \in V_i = [\underline{v}, \overline{v}]$ wins the object with the bid b_i, her utility will be $u(v_i - b_i)$. Types are distributed according to the density function $f : [\underline{v}, \overline{v}]^N \to \mathbb{R}_+$. We have the following:

Theorem 6.1 *Assume that u is twice continuously differentiable, $u' > 0$, $f \in \mathcal{D}^k$, f is symmetric and positive ($f > 0$).*[40] *If $b : [\underline{v}, \overline{v}] \to \mathbb{R}$ is a symmetric monotonic pure strategy equilibrium (SMPSE), then b is continuous in $(0,1)$ and is differentiable almost everywhere in $(0,1)$.*[41] *Moreover, b is the unique symmetric increasing equilibrium. If $u(x) = x^{1-c}$, for $c \in [0,1)$, b is given by*

$$b(x) = x - \int_0^x \exp\left[-\frac{1}{1-c}\int_\alpha^x \frac{f(s \mid s)}{F(s \mid s)} ds\right] d\alpha. \tag{18}$$

Proof. *See the supplement to this paper.* ■

[40] In fact, it is not necessary that f has full support. See the supplement of the paper for details.

[41] b may be non-differentiable only in the points $\frac{m}{k}$, for $m = 0, 1, ..., k$.

Having established the uniqueness of the candidate for equilibrium, our task is reduced to verifying whether this candidate is, indeed, an equilibrium. We begin with the two bidders case and then, generalize it to the n bidders case. Although it is possible to extend our results for risk averse bidders, specially if $u(x) = x^{1-c}$ for $c > 0$, we focus below on the case of risk neutrality $u(x) = x$.

6.1 Two risk neutral players case

Theorem 6.1 establishes the uniqueness of the candidate for symmetric increasing equilibrium for $f \in \mathcal{D}^{\infty} = \cup_{k=1}^{\infty} \mathcal{D}^k$. We have now only to check if the unique candidate is indeed equilibrium. In economics, this is usually done by checking the second order condition. In auction theory, it is more common to appeal to monotonicity arguments based on a single crossing condition (see, among others, Milgrom and Weber (1982) and Athey (2001)). These methods give sufficient conditions for equilibrium, *but these conditions are, in general, not necessary*. Sufficient and necessary conditions would not only provide grounds to understand what really entails equilibrium existence, but also to work with the more general possible setup that yields equilibrium existence. Thus, we take here another approach, which gives *necessary and sufficient* conditions for equilibrium existence. This is done by checking directly the equilibrium conditions, as we explain next.

Let $b(\cdot)$, given by (18) with $c = 0$, denote the candidate for equilibrium. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, z)$ represent the expected interim payoff of a player of type x who bids as a type z, that is, $\Delta(x, z) \equiv \Pi(x, b(z)) - \Pi(x, b(x))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $z \in [0, 1]^2$. In other words, the equilibrium condition requires checking an *inequality for an infinite pair of points*. Of course, it is not possible to check an inequality at an infinite number of points.

If $\Delta(x, z)$ is continuous, then an approximation algorithm could check the inequality only at some points and, with some confidence, ensure equilibrium existence. Of course, this method would not be exact in the sense that approximation errors are inherent to the algorithm. However, for $n = 2$ players, the next theorem shows that when $f \in \mathcal{D}^k$ there is an exact algorithm, that does not introduce errors, and it is fast because it requires only a small number of comparisons.

Theorem 6.2 *Consider Symmetric Risk Neutral Private Value Auction with two players with $f \in \mathcal{D}^{\infty} = \cup_{k \geq 1} \mathcal{D}^k$. There exist an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. For $f \in \mathcal{D}^k$, the algorithm requires less than $3\left(k^2 + k\right)$ comparisons. The algorithm is exact, in the sense that errors can occur only in elementary operations.*[42]

Proof. *See the supplement to this paper.* ■

We should emphasize that Theorem 6.2 gives *necessary and sufficient* conditions for SMPSE existence. This is what we mean by "decides" above. Unfortunately, these necessary and sufficient conditions are long. Theorem 9.5 below contains an explicit statement of those conditions.

The use of the term "algorithm" above should not be confused with a complex procedure for deciding equilibria. On the contrary, the theorem reduces the verification of equilibrium to a set of simple conditions that can be explicitly given. Thus, we can state Theorem 6.2 just through this reduction and even avoid the use of the word "algorithm." This is exactly what Theorem 9.5 does.

Remark 6.3 *It is useful to compare this result with the best algorithms for solving simpler games as bimatrix games (see Savani and Von Stengel (2006)). While best known algorithms for bimatrix games require operations that grow exponentially with the size of the matrix, the number of our comparisons increases with k^2. We do not state that the algorithm runs in polynomial time because our problem is in continuous variables, not in discrete ones. "Polynomial time" would be slightly vague here, since errors of approximations are possible. Nevertheless, as stated, the possible errors are elementary and require a small number of operations. This allows one to realize the important benefits of working with continuous variables but density functions in $\mathcal{D}^k$, as we propose. The characterization of the strategies obtained through differential equations allows one to drastically reduce the computational effort, by reducing the equilibrium candidates to one. The fact that we work on $\mathcal{D}^k$ allows us to precisely characterize a small number of points to be tested for the equilibrium condition. The speed of the method allows auction theorists to run simulations for a big number of trials and get a good figure of what happens in general. From this, conjectures for theoretical results can also be derived.*

[42]By elementary operations we mean sums, multiplications, divisions, comparisons and square roots.

The proof of this theorem is long and complex, because $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and the determination of its maxima for each of these sets. It turns out that we need to check a different number of points (between 1 and 5) for some of these squares. Section 9.3 in the appendix outlines the proof. The complete proof is given in a supplement to this paper.

6.2 Equilibrium results for n players

The ideas in the proof of Theorem 6.2 generalize from 2 to n players with minor complications. Thus, one can use grid distributions (those in $\mathcal{D}^{\infty}$) to study auctions in a more general setup.

As before, the equilibrium candidate is unique and we have an expression for it. Thus, SMPSE will be established if and only if $\Delta(x, z) = \Pi(x, b(z)) - \Pi(x, b(x))$ is non-positive. We can test the signal of $\Delta(x, z)$ for $(x, z) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m, p \in \{1, ..., k\}$. This is simplified to check non-positiveness of a polynomial over $[0, 1]^2$. The only difference from the $n = 2$ case is that in this last case the polynomial is of degree 3 and we can analytically solve it. For $n > 2$, the polynomial (in the two variables, x and z) has a degree of at least $n + 1$ and we have to rely on numerical methods for finding minimal points. The following establishes the existence of an algorithm that solves for this, that only makes errors in numerical approximations:

Theorem 6.4 *Consider symmetric risk neutral private value auction with n players with $f \in \mathcal{D}^{\infty}$. There exists an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. Errors are commited in finding roots of polynomials and in elementary operations.*
Proof. *See the supplement to this paper.* ■

Note that we did not make statements about the speed of the method. This is just because this speed depends on the numerical method used to find roots of polynomials. We were unable to find good characterizations of the running time of solutions to this problem.

Grid distributions are also useful to study *asymmetric auctions*. However, since an explicit expression for $b(\cdot)$ is not available in this case, the above described method needs some adaptation, but the main idea remains the same. Instead of working with explicit solutions that allow us to speed up the algorithm, we can just work with numerical simulations. The difference is that the errors can happen now not only in the roots of polynomials, as in the case above, but also in numerical integrations and the determination of functions of two variables. This difference impacts the speed and accuracy of the method (it is not possible to state a "if and only if" result anymore). However, as long as these numerical problems are well understood, the method does not present particular difficulties.

6.3 The Revenue Ranking of Auctions

As a further illustration of our approach, we show how grid distributions can be used to address the problem of revenue ranking of the first price and second price auctions. Let us denote by R_2^f the expected revenue (with respect to $f \in \mathcal{D}^k$) of the second price auction.[43] Similarly, R_1^f denotes the expected revenue (with respect to $f \in \mathcal{D}^k$) of the first price auction. When there is no need to emphasize the pdf $f \in \mathcal{D}^k$, we write R_1 and R_2 instead of R_1^f and R_2^f. Below, μ refers to the natural measure defined over $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$, as further explained in the supplement to this paper.

The following theorem gives the expression of the expected revenue difference $\Delta_R^f \equiv R_2^f - R_1^f$ between the second and the first price auctions and it is *not* restricted to densities in $f \in \mathcal{D}^\infty$.

Theorem 6.5 *Assume that f has a SMPSE in the first price auction. The revenue difference between the second and the first price auction is given by*

$$\int_0^1 \int_0^x b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \quad (19)$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$.

Proof. *See the appendix.* ■

[43] Remember that, since we are working with private values, second price auctions are equivalent to English auctions.

In order to make a relative comparison, we define $r \equiv \frac{R_2^f - R_1^f}{R_2^f}$, for each f. Generating a uniform sample of $f \in \mathcal{D}^k$, we can obtain the probabilistic distribution of Δ_R^f or of r. The procedure to generate $f \in \mathcal{D}^k$ uniformly is described in the supplement to this paper. The results are shown in subsection 6.4 below.

Moreover, we can also obtain theoretical results about what happens for $\mathcal{D}^k$ for a large k and even for $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$. Nevertheless, for the last case, one has to be careful with the meaning of the "uniform" distribution. In the supplement to this paper we show that a natural measure can be defined for $\mathcal{D}^\infty$, which is analogous to Lebesgue measure, although it cannot have all the properties of the finite dimensional Lebesgue measure.

In this fashion, we are able to obtain previsions based on simulations and also theoretical results. One possible objection to this approach is that it considers too equally the pdf's in the sets $\mathcal{D}^k$. But this is just because we are not assuming any specific information about the context where the auction runs—in some sense, this is a "context-free"approach. If one has information on the environment where the auction runs, so that one can restrict the set of suitable pdf's, then the uniform measure should be substituted by the empirical measure obtained from this environment. Obviously, the method can be easily adapted to this, once one has such "empirical measure" of the possible distributions.

Now, we present the results that one can obtain using this approach.

6.4 Numerical results

The method developed in this paper allows us to compare the revenue ranking of first and second price auction with general dependence. In the figure below, we show the histograms of $r \equiv \frac{R_2^f - R_1^f}{R_2^f}$ where $f \in \mathcal{D}^k$ is drawn from a uniform distribution among the distributions in $\mathcal{D}^k$, for $k = 3, ..., 10$. By this we mean that each $f \in \mathcal{D}^k$ is equally likely to be drawn in our simulation. The results remain qualitatively the same if, instead of the uniform distribution in $\mathcal{D}^k$, we choose some normal distribution in $\mathcal{D}^k$ with peak centered around the independent and symmetric $f \in \mathcal{D}^k$ (i.e., $f(x, y) \equiv 1, \forall (x, y)$).[44] In the figure, two curves are shown in the graph of each k: one corresponds

[44]We use the euclidean metric for defining a distance between densities in $\mathcal{D}^k$. Then, we use this distance from the $f \equiv 1$ as the parameter of a unidimensional normal with zero average and some positive variance. We tested many variances and the results do not change much. When the variance is big, the numbers become very similar to the previous case, where all distributions in $\mathcal{D}^k$ are equally likely.

to the histogram for all f such that there is a SMPSE in the first price auction; the other curve corresponds to all f such that there is a SMPSE in the first price auction *and* the correlation implied by f is non-negative. As the reader can see, this restriction does not change significantly the results, that is, the difference between the two histograms is small. Although we do not include the graphics here, the results are also stable for $n > 2$.

6.5 Approximation

Although this is not our preferred view of our approach, one could see it as an approximation, especially in the context of grid distributions.

Theorem 6.6 *The set of grid distributions is dense in $\Delta(V)$, endowed with its natural (weak*) topology. Also, the set $\mathcal{D}^\infty$ of grid densities is dense in:*

(i) $\mathcal{D}$, the set of densities $f : T \to \mathbb{R}_+$, endowed with the pointwise topology.

(ii) L^p, the set $\mathcal{D}$ endowed with the L^p-norm (w.r.t. Lebesgue measure).

(iii) $\mathcal{C}$, the set of continuous densities, endowed with the sup-norm.[45]

Theorem 6.6 shows that by restricting the set of distributions on types that we consider to grid distributions, we do not lose much: we can approximate any measure as well as we want. An immediate corollary of this result is that an econometrician cannot reject grid distributions with any amount of finite data.

The transformation $\mathbb{T}^k$ is stable for some conditions like affiliation.[46] Indeed, let $\mathcal{A} \subset \mathcal{D}$ denote the set of affiliated density functions; we have the following:

Proposition 6.7 *Assume that f is continuous. Then f is affiliated if and only if for all k, $\mathbb{T}^k(f)$ also is. In mathematical notation: $f \in \mathcal{A} \Leftrightarrow \mathbb{T}^k(f) \in \mathcal{A}$, $\forall k \in \mathbb{N}$ or yet, $\mathcal{A} = \cap_{k\in\mathbb{N}} \left(\mathbb{T}^k\right)^{-1} \left(\mathcal{A} \cap \mathcal{D}^k\right)$.*

More importantly, the approximation given by $\mathbb{T}^k$ preserves pure strategy equilibria for all continuous Bayesian games.[47]

[45]This is a slight abuse of terminology, since that $\mathcal{D}^\infty$ is not contained in $\mathcal{C}$. What we show is that for each $f \in \mathcal{C}$ and $\epsilon > 0$, there exists $k \in \mathbb{N}$ and $f^k \in \mathcal{D}^k$ such that $\|f - f^k\| \equiv \sup_t |f(t) - f^k(t)| < \epsilon$.

[46]We say that $f \in \mathcal{D}$ is affiliated if for all $x, y \in T$, $f(x)f(y) \leqslant f(x \wedge y)f(x \vee y)$.

[47] Whenever discontinuities occur in zero probability sets, as in the private value auctions covered by Theorem 5.1, the conclusion of Propositions 6.8 and 6.9 still hold.

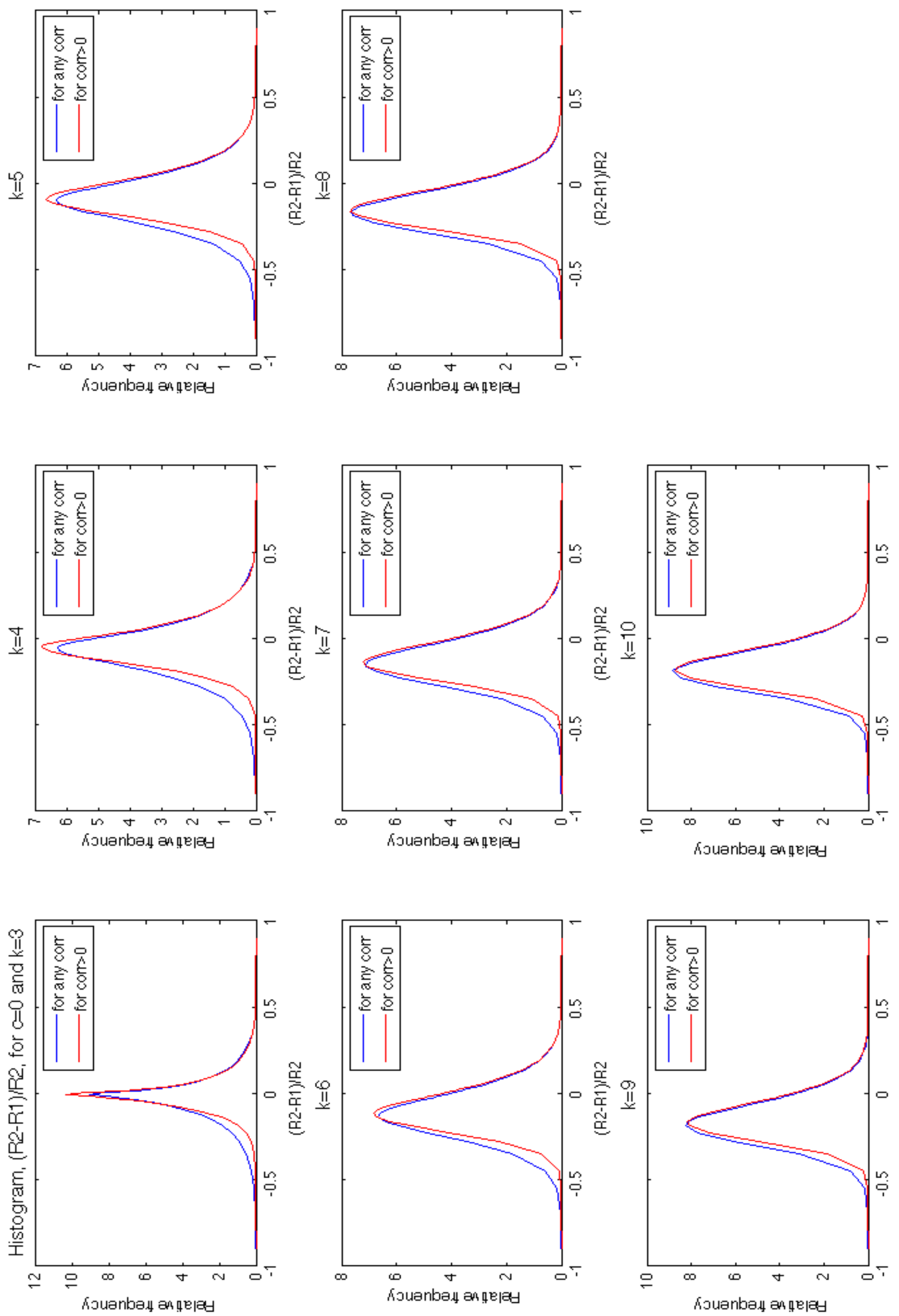

Figure 1: Histogram of Revenue Differences

Proposition 6.8 *Assume that f and the ex post utility $u_i : T \times A \to \mathbb{R}$ are continuous, $\forall i \in I$. Assume that there is a sequence $\{k_n\}$ such that $\mathbb{T}^{k_n}(f)$ has an equilibrium $s^{k_n} = (s_1^{k_n}, ..., s_N^{k_n})$ for all n and the sequence $\{s^{k_n}\}$ converges pointwise to $s \in \mathcal{F}$. Then, s is a pure strategy equilibrium under f.*

Although a complete converse of the above result is not possible, because the equilibrium inequality can be approximated by above, the following provides a partial converse.

Proposition 6.9 *Let $f \in \mathcal{D}$ have an equilibrium $s \in \mathcal{F}$ and assume that u_i is continuous for every i. Then for each $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that s is a ε-equilibrium for $\mathbb{T}^k(f)$, for all $k \geqslant k_\varepsilon$.*

The above result is satisfactory for numerical applications, because numerical calculation will involve errors anyway. Thus, in this case the above restriction is innocuous.

7 Discussion

This section discusses the relation of our work to existing literature. We begin by discussing Neemans's "beliefs do not determine preferences" in section 7.1. Section 7.2 comments on the differences and similarities with Athey-McAdams-Reny's approach to establishing equilibrium existence in Bayesian games. Other papers are discussed in section 7.3.

7.1 Beliefs do not determine preferences

Neeman (2004) and Heifetz and Neeman (2006) study the circumstances under which a seller can extract full surplus from informed buyers with correlated types, as demonstrated by Crémer and McLean (1985).

They show that full extraction of surplus requires that "beliefs determine preferences." That is, it requires a setting where almost all types have different beliefs; if types share beliefs, full extraction is not possible. This suggests that models where beliefs do not determine preferences (non-BDP) are more realistic and desirable from a mechanism design point of view. This paper argues essentially the same, but from the point of view of applications to general Bayesian games.

It should be noted that our $(*)$ is stronger than their non-BDP in two aspects. First, while it is enough for them that a positive mass of types share beliefs, we require that every set with a positive mass contains types sharing beliefs. Second, $(*)$ requires that types are strictly ordered, while they do not need any order on types. Despite these technical differences, which are necessary to establish our results, we view the general message of both works as very similar: we should give more attention to models where beliefs do not determine preferences.

The economic intuition for why this is reasonable was given by Neeman (2004, p. 67-8) in the following terms:

> Issues of generality notwithstanding, preferences and beliefs have traditionally been considered to be independent of one another in both economic and decision theory. This tradition presumably reflects the idea that the processes that generate utilities and beliefs are cognitively distinct and causally "independent," or at least should be treated as such.

There is a natural way of interpreting that preferences and beliefs are "independent," as Neeman puts it: that types are formed as in a product structure of preferences and beliefs. In this interpretation, one should require that the measure on types is absolutely continuous with respect to the product of the marginals over preferences and types. As established by Lemma 2.3, this implies our $(*)$. Therefore, $(*)$ also captures their intuition.

7.2 Relation to Athey-McAdams-Reny's approach

Athey (2001), McAdams (2003), Reny and Zamir (2004) and Reny (2011) consider the best reply of each player to *monotonic* strategies played by the opponents. Since they are interested in proving only the *existence* of monotonic equilibria, this is enough and convenient. They work with different (increasingly weaker) assumptions on *interim* payoff functions when the opponents play monotonic strategies and use different forms of single crossing conditions to obtain their equilibrium results. Their conditions are then checked on specific games, leading to pure strategy equilibrium results for those games. Assumptions on distributions are in general not necessary at the interim level, so they appear only at the examples given. An exception is Reny and Zamir (2004), who mainly consider first price interdependent values auctions with n bidders and affiliated types, while Athey (2001) had considered the case of $n = 2$ bidders. With respect to multi-unit auctions, all of these papers contain results assuming that types are independent.

Since their general model is on the interim stage and no correlation assumption is needed at this level, these authors seem to aim results that do not require independence. Therefore, it is very understandable that they did not consider properties that seem to crucially depend on independence. Namely, they did not explore the property that if types are independent, then a best reply to *any strategy*—not only to monotonic ones—needs to be monotonic (under standard monotonicity assumptions on the utility functions). This property was first observed for first price auction by Maskin and Riley (2000) and later generalized by Araujo and de Castro (2009). Theorem 4.1, which generalizes this result, seems to be the first of its kind for general action spaces.[48] Note that the property leads to a much stronger kind of result: every equilibrium, if it exists, is in pure strategies. It seems that the fact that these authors, not being interested primarily on independence, did not consider such kind of implication.

What is perhaps more surprising is that a general approach to dependence can be worked out from independence ideas. The key is to look at beliefs, as $(*)$ clarifies. However, even if the strategies are not overall monotonic,[49] comparative statics results are possible, in the same fashion as the case of monotonic strategies.[50]

This still opens the question of when an actually monotonic equilibrium exists. Note that this question is not very important for the previous authors, because they were interested in providing *sufficient* conditions for the existence of a monotonic strategy. With the results of this paper, we know that a pure strategy equilibria always exists; the question now is whether there are only non-monotonic strategies equilibria or maybe there is a monotonic strategy equilibria. In a particular setting, we are able to provide a complete characterization of the existence of monotonic equilibria. Since this setting is now a special case of the standard approach of unidimensional types, it allows us to answer a question that has not been tackled before by the literature.

7.3 Related literature and conclusion

In this paper we advocated for the use a special class of distributions that allows general dependence and asymmetry in general games with incom-

[48]The generalization for infinite-dimensional action spaces is not trivial. Among other things, it requires some form of restriction on the metric space considered, as Assumption 3.2. Also, our argument is quite different from previous results.

[49]Note that the pure strategy equilibria shown to exist for private values auctions can be considered monotonic in the order defined. See Remark 5.2.

[50]Space limitations refrain us from detailing further comments on this, but the reason for why comparative statics is as easy (or difficult) as before, is that we also work with orders and obtain (constrained) monotonic equilibria.

plete information. This class of distribution is more general than normally considered ones, but can be treated both with theoretical and numerical methods. We illustrated the potential applications with theoretical results and computer experiments (simulations). It is shown that a fast algorithm exists for determining symmetric pure strategy equilibrium existence in auctions with n players. We also proved the existence of pure strategy equilibrium in Bayesian games with a new monotonicity condition.

The grid distributions discussed here appeared first in de Castro (2008) and were used by de Castro and Paarsch (2010) to test for affiliation.

Fang and Morris (2006) study the revenue ranking of first and second price auctions in a model with finite correlated types, who share some beliefs. They obtain equilibrium in mixed strategies because they work with finite types, but one of the characterizations of the equilibrium (monotonicity of best replies) could be obtained using our Theorem 4.1. Notice that they also write the types as consisting of two parts: the preference part v_i and the belief part δ_i (in our notation).

Although we focused mainly (but not exclusively) on equilibrium existence results, so that most of our results are deeply related to equilibrium existence papers like Athey (2001), McAdams (2003), Jackson and Swinkels (2005) and Reny (2011), our main contribution is to offer a new framework for working with correlation of types. This contribution can also be explored in applied works.

Appendix

8 Appendix A: Main Proofs

8.1 Proofs for Sections 2.1 and 3.1

8.1.1 Proofs for Section 2.1

Proof of Lemma 2.1: Given a measurable $E \subset V_i$ satisfying $\gamma_i(E) > 0$, for each $\delta \in \Delta_i$ define $E^\delta \equiv \{v \in E : \delta \in \Gamma(v)\}$. Let $E' \equiv \cup_{\delta \in \Delta_i'} E^\delta \subset \cup_{\delta \in \Delta_i} E^\delta = E$. Since the set $V_i' = \cup_{\delta \in \Delta_i'} V_i^\delta$ has full measure by (ii), $\gamma_i(E') = \gamma_i(E \cap V_i') > 0$. Since Δ_i' is countable and $\sum_{\delta \in \Delta_i'} \gamma_i(E^\delta) \geqslant \gamma_i(E') > 0$, for at least one $\delta \in \Delta_i'$, we have $\gamma_i(E^\delta) > 0$. By (i), there exist $v, v' \in E^\delta$ such that $v' \succ v$. Since $\delta \in \Gamma(v) \cap \Gamma(v')$, there exist t_i and t_i' such that $\widehat{V}_i(t_i) = v, \widehat{\delta}_i(t_i) = \delta$, $\widehat{V}_i(t_i') = v'$ and $\widehat{\delta}_i(t_i') = \delta$. By the order definition, (1), the existence of these t_i, t_i' establishes $(*)$. ∎

Proof of Lemma 2.3: For a set $E \in \mathcal{T}_i$, let E_δ denote projection of E over V_i, that is, $E_\delta \equiv \{v \in V_i : (v, \delta) \in m(E)\}$. Let π_i denote the product measure $\gamma_i \times \nu_i$. By Fubini's Theorem, $\pi_i(m(E)) = \int_{\Delta_i} \gamma_i(E_\delta)\nu_i(\delta)$. Thus, if $\tau_i(E) > 0$, which implies $\pi_i(m(E)) > 0$, the set of $\delta \in \Delta_i$ such that $\gamma_i(E_\delta) > 0$ has ν_i-positive measure. Fix any δ in this set. Since $\gamma_i(E_\delta) > 0$, by condition (i) of Lemma 2.1, there exist $v, v' \in E_\delta$ such that $v' \succ v$. By the definition of $\geqslant_i$, we have that $t_i \equiv (v, \delta), t_i' \equiv (v', \delta)$ satisfy $(*)$. ∎

An inspection of the above proof shows that we actually need less than absolute continuity with respect to $\pi_i = \gamma_i \times \nu_i$. It is enough that $\tau_i(E) > 0$ imply the existence of a δ such that $\gamma_i(E_\delta) > 0$. But this is essentially just a restatement of $(*)$ for this particular setting with a product structure.

8.1.2 Proofs for Section 3.1

It is useful to recall some definitions.

Definition 8.1 (Supermodularity) *g is supermodular in Y if for any $x \in X$ and $z \in Z$ and any pair $y, y' \in X$, we have:*

$$g(x, y, z) - g(x, y \wedge y', z) \quad \leqslant \quad g(x, y \vee y', z) - g(x, y', z).$$

Also, we say that g is strictly supermodular if the inequality above is strict whenever x and x' are incomparable.

It is well-known and easy to verify that supermodularity is preserved under integration. It is also easy to see that supermodularity implies quasi-supermodularity:

Definition 8.2 (Quasi-supermodularity) *Assume that Y is a lattice and consider the following implication, implicitly supposed to hold $\forall x \in X, \forall y, y' \in Y, \forall z \in Z$:*

$$g(x, y, z) \ \mathbf{R} \ \ g(x, y \wedge y', z) \quad \Longrightarrow \quad g(x, y \vee y', z) \ \mathbf{R}' \ \ g(x, y', z). \tag{20}$$

We say that g is:

- *weak quasi-supermodular in Y if (20) holds with $(\mathbf{R}, \mathbf{R}') = (\geqslant, \geqslant)$;*
- *strictly quasi-supermodular in Y if (20) holds with $(\mathbf{R}, \mathbf{R}') = (\geqslant, >)$;*
- *quasi-supermodular in Y if (20) holds both with $(\mathbf{R}, \mathbf{R}') = (\geqslant, \geqslant)$ and $(\mathbf{R}, \mathbf{R}') = (>, >)$.*

Although x does not a play a role in supermodularity and quasi-supermodularity, it is important for nondecreasing differences and single-crossing properties.

Definition 8.3 (Nondecreasing differences) *g satisfies the non-decreasing differences property in $X \times Y$ if for any $x, x' \in X$ $y, y' \in Y'$ and $z \in Z$, such that $x < x'$ and $y < y'$,*

$$g(x, y', z) - g(x, y, z) \leqslant g(x', y', z) - g(x', y, z). \tag{21}$$

It satisfies increasing differences in $X \times Y$ if (21) holds with $<$ instead of $\leqslant$.

As it is well known (and easy to see), the nondecreasing differences implies single-crossing, while increasing differences implies strict single-crossing.

Definition 8.4 (Single-crossing property) *Consider the following implication, supposed to hold for any $x, x' \in X$ $y, y' \in Y'$ and $z \in Z$, such that $x < x'$ and $y < y'$:*

$$g(x, y', z) \ \mathbf{R} \ g(x, y, z) \implies g(x', y', z) \ \mathbf{R}' \ g(x', y, z); \tag{22}$$

We say that g satisfies:

- *weak single-crossing in $X \times Y$ if (22) holds with $(\mathbf{R}, \mathbf{R}') = (\geqslant, \geqslant)$;*
- *strict single-crossing in $X \times Y$ if (22) holds with $(\mathbf{R}, \mathbf{R}') = (\geqslant, >)$;*
- *single-crossing in $X \times Y$ if (22) holds both with $(\mathbf{R}, \mathbf{R}') = (\geqslant, \geqslant)$ and $(\mathbf{R}, \mathbf{R}') = (>, >)$.*

Proof of Proposition 3.2: Let $x, x' \in X$, such that $x < x'$; $y, y' \in Y$ such that $\neg(y' \geqslant y)$, and $z \in Z$. Since Y is a lattice, there exists $\overline{y} \equiv y \vee y'$ and $\underline{y} \equiv y \wedge y'$. From supermodularity, we have:

$$g(x, y, z) - g(x, \underline{y}, z) \quad \leqslant \quad g(x, \overline{y}, z) - g(x, y', z). \tag{23}$$

Since $\neg(y' \geqslant y)$, $\overline{y} \neq y'$, that is , $\overline{y} > y'$. By increasing differences, we have

$$g(x, \overline{y}, z) - g(x, y', z) < g(x', \overline{y}, z) - g(x', y', z). \tag{24}$$

Then, (23) and (24) imply (8). In the case of strict supermodularity and non-decreasing differences, (23) holds with a strict inequality sign while (24) holds with weak inequality, still giving (8). ∎

Proof of Proposition 3.4: Let $x, x' \in X$, such that $x < x'$; $y, y' \in Y$ such that $\neg(y' \geqslant y)$, and $z \in Z$. Since Y is a lattice, there exists $\overline{y} \equiv y \vee y'$ and $\underline{y} \equiv y \wedge y'$. From weak quasi-supermodularity, we have:

$$g(x, y, z) \geqslant \; g(x, \underline{y}, z) \quad \Longrightarrow \quad g(x, \overline{y}, z) \geqslant \; g(x, y', z). \tag{25}$$

Since $\neg(y' \geqslant y)$, $\overline{y} > y'$. By strict single crossing, we have

$$g(x, \overline{y}, z) \geqslant \; g(x, y', z) \quad \Longrightarrow \quad g(x', \overline{y}, z) > \; g(x', y', z); \tag{26}$$

Then, (25) and (26) imply (9). In the case of strict quasi-supermodularity and single crossing, (25) holds with a strict inequality sign on the left, while (26) holds with strict inequalities in both sides, still giving (9). ∎

Recall that S_{-i} is the set of behavioral strategies by players $j \neq i$. Fix $\delta \in \Delta_i \equiv \widehat{\delta}_i(T_i)$ and define $T_{i\delta} \equiv \{t_i \in T_i : \widehat{\delta}_i(t_i) = \delta\}$.

Lemma 8.5 *If $u_i : T_{i\delta} \times A_i \times T_{-i} \times A_{-i} \to \mathbb{R}$ satisfies the ID-monotonicity property in $T_{i\delta} \times A_i$, so does $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$.*

Proof. Consider $t_i, t_i' \in T_{i\delta}$ such that $t_i < t_i'$, and $a_i, a_i' \in A_i$ such that $\neg(a_i' \geqslant a_i)$. Then, there exists $\underline{a}_i, \overline{a}_i \in A_i$ such that $\overline{a}_i \neq a_i'$ and

$$u_i(t_i, a_i, \cdot) - u_i(t_i, \underline{a}_i, \cdot) < u_i(t_i', \overline{a}_i, \cdot) - u_i(t_i', a_i', \cdot), \tag{27}$$

where "$\cdot$" stands for t_{-i}, a_{-i}. Now, integrating with respect to a mixed strategy $\mu_{-i} \in S_{-i}$ and the belief $\delta \in \widehat{\delta}_i(T_i) \subset \Delta(T_{-i})$, we obtain:

$$\begin{aligned}
& \Pi(t_i, a_i) - \Pi(t_i, \underline{a}_i) \\
= & \int_{T_{-i}} \int_{A_{-i}} \left[u_i(t_i, a_i, t_{-i}, a_{-i}) - u_i(t_i, \underline{a}_i, t_{-i}, a_{-i})\right] \mu_{-i}(t_{-i}, da_{-i}) \delta(dt_{-i}) \\
< & \int_{T_{-i}} \int_{A_{-i}} \left[u_i(t_i', \overline{a}_i, t_{-i}, a_{-i}) - u_i(t_i', a_i', t_{-i}, a_{-i})\right] \mu_{-i}(t_{-i}, da_{-i}) \delta(dt_{-i}) \\
= & \Pi(t_i', \overline{a}_i) - \Pi(t_i', a_i'),
\end{aligned} \tag{28}$$

as we wanted to show. ∎

Eventually, we want to establish the monotonicity property of Π_i without having the exact monotonicity property on u_i. The following Lemma can be useful in this case.[51]

Lemma 8.6 *Assume that* $u_i : T_{i\delta} \times A_i \times T_{-i} \times A_{-i} \to \mathbb{R}$ *satisfies the following:*

(i) it is supermodular in A_i*;*

(ii) it satisfies non-decreasing differences in $T_{i\delta} \times A_i$*;*

(iii) for any $t_i^1, t_i^2 \in T_{i\delta}$ *and* $a_i^1, a_i^2 \in A_i$ *and* $\mu_{-i} \in S_{-i}$ *satisfying* $t_i^1 < t_i^2$, $a_i^1 <_i a_i^2$, *we have* $\delta(T'_{-i}) > 0$, *where* T'_{-i} *is the set defined by those* $t_{-i} \in T_{-i}$ *for which*

$$\begin{aligned} &\int_{A_{-i}} \left[u_i(t_i^1, a_i^2, t_{-i}, a_{-i}) - u_i(t_i^1, a_i^1, t_{-i}, a_{-i})\right] \mu_{-i}(t_{-i}, da_{-i}) \\ < \ &\int_{A_{-i}} \left[u_i(t_i^2, a_i^2, t_{-i}, a_{-i}) - u_i(t_i^2, a_i^1, t_{-i}, a_{-i})\right] \mu_{-i}(t_{-i}, da_{-i}). \end{aligned}$$

Then $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ *has the ID-monotonicity property in* $T_{i\delta} \times A_i$.

Proof. Using (i) and (ii), we can repeat the proof of Proposition 3.2, where both (23) and (24) are with weak inequalities. Now, following the proof of Lemma 8.5, we have (27) holding with weak inequality, which also implies (28) with weak inequality. However, because of (iii), (28) actually holds with strict inequality. This concludes the proof. ∎

8.2 Proofs for Section 4

8.2.1 Proof of Theorems 4.1 and 4.2

By Lemma 8.5, ID-monotonicity on u_i implies ID-monotonicity on Π_i which trivially implies SC-monotonicity on Π_i. Thererfore, Theorem 4.1 is implied by Theorem 4.2 and it is enough to prove this last one.

The proof will be divided in a number of lemmas. In all results below, we fix $\delta \in \Delta_i \equiv \widehat{\delta}_i(T_i)$ and define $T_{i\delta} \equiv \{t_i \in T_i : \widehat{\delta}_i(t_i) = \delta\}$.

Lemma 8.7 *Assume that* Π_i *satisfies the SC-monotonicity property in* $T'_i \times A_i$, *where* $T'_i \subset T_i$. *Let* $BR_i : T'_i \to A_i$ *denote the correspondence of* i*'s best replies to* μ_{-i}. *Then any selection of* BR_i *is monotone nondecreasing, that is, for any* $t_i, t'_i \in T_i$ *such that* $t'_i > t_i$ *and* $a_i \in BR_i(t_i)$, $a'_i \in BR_i(t'_i)$, *we have* $a'_i \geqslant a_i$.

[51] It is actually used in our proof of Theorem 5.1.

Proof. Let $t_i, t_i' \in T_i$ be such that $t_i' > t_i$ and let $a_i \in BR_i(t_i)$ and $a_i' \in BR_i(t_i')$. We want to prove that $a_i' \geqslant a_i$. Suppose otherwise. By the SC-monotonicity property, there exist $\underline{a}_i, \overline{a}_i \in A_i$ safisfying

$$\Pi_i(t_i, a_i) \geqslant \Pi_i(t_i, \underline{a}_i) \text{ implies } \Pi_i(t_i', \overline{a}_i) > \Pi_i(t_i', a_i').$$

Since $a_i \in BR_i(t_i)$, the first inequality holds, which implies $\Pi_i(t_i', \overline{a}_i) > \Pi_i(t_i', a_i')$, contradicting that $a_i' \in BR_i(t_i')$. This concludes the proof. ∎

The following technical result about chains will be used below.

Lemma 8.8 *Let C be a chain in a partially ordered set $(X, \geqslant)$ with at least three elements. Then there exist disjoint sets C_1, C_2 and C_3 such that $C_1 \cup C_2 \cup C_3 = C$ and for any two points $x, y \in C_i$ with $x > y$, for $i = 1, 2, 3$, there exists $z \in C \setminus C_i$ such that $x > z > y$.*

Proof. Let $\mathcal{E}$ denote the class of pair of sets (C_1, C_2) such that $C_1 \cap C_2 = \emptyset$, $C_1, C_2 \subset C$ and satisfying the following:

($\bullet$) for $i = 1, 2$ and any $x, y \in C_i$ with $x > y$, there exists $z \in C_{3-i}$ such that $x > z > y$.

Order $\mathcal{E}$ by inclusion, that is, $(E_1, E_2) \succcurlyeq (C_1, C_2)$ if $E_i \supset C_i$ for $i = 1, 2$.

Consider a chain $\{(C_1^\lambda, C_2^\lambda)_{\lambda \in \Lambda}\}$ in $\mathcal{E}$. Define $D_1 = \cup_{\lambda \in \Lambda} C_1^\lambda$ and $D_2 = \cup_{\lambda \in \Lambda} C_2^\lambda$. We claim that $(D_1, D_2) \in \mathcal{E}$. Indeed, trivially $D_1, D_2 \subset C$. If there exists $x \in D_1 \cap D_2$, then $x \in C_1^\lambda$ and $x \in C_2^{\lambda'}$ for some $\lambda, \lambda' \in \Lambda$, but since Λ determines a chain, either $(C_1^\lambda, C_2^\lambda) \succcurlyeq (C_1^{\lambda'}, C_2^{\lambda'})$ or $(C_1^{\lambda'}, C_2^{\lambda'}) \succcurlyeq (C_1^\lambda, C_2^\lambda)$. Thus, we have either $x \in C_1^\lambda \cap C_2^\lambda \supset C_1^\lambda \cap C_2^{\lambda'}$ or $x \in C_1^{\lambda'} \cap C_2^{\lambda'} \supset C_1^\lambda \cap C_2^{\lambda'}$, but this contradicts $C_1^\lambda \cap C_2^\lambda = C_1^{\lambda'} \cap C_2^{\lambda'} = \emptyset$. Thus, $D_1 \cap D_2 = \emptyset$. Finally, we observe that (D_1, D_2) satisfies ($\bullet$). Indeed, fix $i = 1$ or 2 and pick $x, y \in D_i$ with $x > y$. Similarly to the previous argument, there exists $\lambda \in \Lambda$, such that $x, y \in C_i^\lambda$. But because $(C_1^\lambda, C_2^\lambda) \in \mathcal{E}$, then there exists $z \in C_{3-i}^\lambda$ such that $x > z > y$. But then $z \in D_{3-i}$, which completes the proof of the claim.

Trivially, (D_1, D_2) is an upper bound for the chain $\{(C_1^\lambda, C_2^\lambda)_{\lambda \in \Lambda}\}$. Therefore, by the Zorn's Lemma, there exists a maximal element (C_1, C_2) on $\mathcal{E}$. We will need to establish some facts about such maximal element.

Suppose that there exists $x \in C \setminus (C_1 \cup C_2)$. Since (C_1, C_2) is maximal in $\mathcal{E}$, we must have $(C_1 \cup \{x\}, C_2) \notin \mathcal{E}$ and $(C_1, C_2 \cup \{x\}) \notin \mathcal{E}$. This means that ($\bullet$) fails for both pairs, that is,

$(\star)_i$: $\exists y_i \in C_i$ such that $x > y_i$ and $C_{3-i} \cap [y_i, x] = \emptyset$ or $y_i > x$ and $C_{3-i} \cap [x, y_i] = \emptyset$[52]

[52] Here, $[w, z] \equiv \{u \in X : w \leqslant u \leqslant z\}$. Observe that since $y_1, x \notin C_2$, $C_2 \cap [y_1, x] = C_2 \cap (y_1, x)$, where $(w, z) \equiv \{u \in X : w < u < z\}$.

holds for both $i = 1$ and $i = 2$. Fix $y_1 \in C_1$ and $y_2 \in C_2$ satisfying $(\star)_1$ and $(\star)_2$ respectively. Since C is a chain, we may assume, without loss of generality, that $y_1 > y_2$. If $x > y_1 > y_2$ then $(\star)_2$ would be false; therefore, we must have $y_1 > x$. Similarly, if $y_2 > x$, we would have $y_1 > y_2 > x$, contradicting $(\star)_1$. Therefore, we must have $y_1 > x > y_2$. Observe that $C_2 \cap (x, y_1) = \emptyset$ and $C_1 \cap (y_2, x) = \emptyset$. Also, $C_1 \cap (x, y_1) = \emptyset$; otherwise, let $y' \in C_1 \cap (x, y_1)$. Since $(C_1, C_2) \in \mathcal{E}$, by $(\bullet)$ there would exist $z \in C_2$ such that $y_1 > z > y'$ which would imply $z \in C_2 \cap (x, y_1)$, an absurd. Similarly, we have $C_2 \cap (y_2, x) = \emptyset$. Therefore, $(C_1 \cup C_2) \cap (y_2, y_1) = \emptyset$. Actually, we claim that more is true, namely, $C \cap (y_2, y_1) = \{x\}$.

To see this, suppose that $x, x' \in C \cap (y_2, y_1)$ for some $x' \neq x$. Since $(C_1 \cup C_2) \cap (y_2, y_1) = \emptyset$, it must be the case that $x' \in C \setminus (C_1 \cup C_2)$. Assume $x' > x$—the case $x > x'$ is analogous, switching x and x'. We will prove that $(C_1', C_2') \equiv (C_1 \cup \{x\}, C_2 \cup \{x'\}) \in \mathcal{E}$. To see this, it is enough to verify $(\bullet)$. Indeed, let $u, v \in C_1'$ be such that $u > v$. If $u, v \in C_1$ there is nothing to prove. If $u = x$, then $y_2 > v$, which implies that $x > y_2 > v$, that is, $(\bullet)$ is satisfied. On the other hand, if $v = x$, then $u \geqslant y_1 > x' > x$ with $x' \in C_2'$ and $(\bullet)$ is also satisfied. The argument for $u, v \in C_2'$ is analogous. Therefore, $(C_1', C_2') \in \mathcal{E}$, but this contradicts the fact that (C_1, C_2) is maximal on $\mathcal{E}$. Therefore, $C \cap (y_2, y_1) = \{x\}$.

Let $\{x^\lambda\}_{\lambda \in \Lambda}$ denote the set $C_3 \equiv C \backslash (C_1 \cup C_2)$. By the above reasoning, for each $x^\lambda \in C_3$, there exist $\underline{y}^\lambda$ and $\overline{y}^\lambda$ such that (i) $\overline{y}^\lambda > x > \underline{y}^\lambda$; (ii) $C \cap (\underline{y}^\lambda, \overline{y}^\lambda) = \{x^\lambda\}$ and (iii) $\underline{y}^\lambda \in C_i$ and $\overline{y}^\lambda \in C_{3-i}$ for either $i = 1$ or $i = 2$.

Let $x^\lambda, x^{\lambda'} \in C_3$ be such that $x^{\lambda'} > x^\lambda$. Thus, $x^{\lambda'} > \underline{y}^{\lambda'} \geqslant \overline{y}^\lambda > x^\lambda$. Therefore, there exists $\overline{y}^\lambda \in C_1 \cup C_2$ between $x^{\lambda'}$ and x^λ. This shows that the C_1, C_2 and C_3 just defined satisfy the required property. ∎

Given a set $S \subset A_i$, its diameter will be denoted $diam(S)$ and defined by:

$$diam(S) \equiv \sup_{a_i, a_i' \in S} \rho_i(a_i, a_i').$$

For each $n, j \in \mathbb{N}$ and $i \in I$, let $T_{i\delta}^n$ denote the set $\{t_i \in T_{i\delta} : diam\,(BR_i(t_i)) > \frac{1}{n}\}$. Recall that a set $C \subset \mathbb{R}^\ell$ is a chain if $a, b \in C$ implies $a \geqslant b$ or $b > a$. Also, $|C|$ denotes the number of elements of the set C, with $|C| = \infty$ if C is not finite.

Lemma 8.9 *There is a number L_n such that if $C \subset T_{i\delta}^n$ is a chain, then C has at most L_n elements.*[53]

Proof. Let us denote the elements of the chain $C \subset T_{i\delta}^n$ by $\{t_i^k\}_{k \in K}$, where K is an arbitrary set. For each $k \in K$, fix a pair $a_i^k, \tilde{a}_i^k \in BR_i(t_i^k)$ such that

$$\rho_i(a_i^k, \tilde{a}_i^k) > \frac{1}{n}. \tag{29}$$

[53] Note that this is stronger than to say that any chain has a finite number of elements. We need the stronger claim, in order to apply Lemma 8.10 below.

If C has less than three elements, there is nothing to prove. Otherwise, by Lemma 8.8, we have three disjoint sets K_1, K_2 and K_3 such that $K_1 \cup K_2 \cup K_3 = K$ and for $j = 1, 2$ and any $k, k' \in K_j$ such that $t_i^{k'} > t_i^k$, there exists $k'' \in K \setminus K_j$ such that $t_i^{k'} > t_i^{k''} > t_i^k$. In this case, by Lemma 8.7, $a_i^k, \tilde{a}_i^k \leqslant a_i^{k''}, \tilde{a}_i^{k''} \leqslant a_i^{k'}, \tilde{a}_i^{k'}$.[54] Therefore, by Assumption 3.2 and (29), we have $\rho_i(a_i^k, a_i^{k'}) > \frac{1}{n}$; $\rho_i(\tilde{a}_i^k, a_i^{k'}) > \frac{1}{n}$; $\rho_i(a_i^k, \tilde{a}_i^{k'}) > \frac{1}{n}$ and $\rho_i(\tilde{a}_i^k, \tilde{a}_i^{k'}) > \frac{1}{n}$. That is, any point in the set $B_j \equiv \cup_{k \in K_j} \{a_i^k, \tilde{a}_i^k\}$, for $j = 1, 2, 3$, has a distance of at least $\frac{1}{n}$ to any other point in the same set.

Now, consider the open cover of A_i by balls of radius $\frac{1}{2n}$, with center in each of the points of A_i. Since A_i is ρ_i-compact by Assumption 3.1, it is covered by just $\ell_n \in \mathbb{N}$ of these balls. Since $\rho_i(x, y) > \frac{1}{n}$ for any $x, y \in B_j$, $j = 1, 2, 3$, there are no two points of B_j in the same ball. Therefore, there are at most ℓ_n points in each B_j, which shows that there are at most $L_n = 3\ell_n$ points in K. ∎

The height of a partially ordered set (poset) $(X, \geqslant)$ is the number of elements in the highest chain contained in X. Recall that a set A is an antichain if $a, b \in A$ implies that $a \not\geqslant b$ and $b \not\geqslant a$ (of course, it could be $a = b$). The following result is due to Dilworth (1950); for this version of the result, see Trotter (1992).

Lemma 8.10 (Dilworth's Theorem) *If $(X, \geqslant)$ is a poset with height m, then there exists a partition $X = A_1 \cup ... \cup A_m$, where A_i is an antichain for $i = 1, ..., m$.*

Proof of Theorem 4.2: The first part of Theorem 4.2 was proved in Lemma 8.7. By Lemma 8.9, for each δ and n, $T_{i\delta}^n$ has height L_n. By Lemma 8.10, $T_{i\delta}^n$ is formed by the union of L_n antichains. The set of those $t_i \in T_{i\delta}$ that can possibly play mixed strategies, that is, those t_i for which $BR(t_i)$ contains more than one point is contained in $\cup_{n \in \mathbb{N}} T_{i\delta}^n$ and, therefore, this is a denumerable union of antichains. ∎

8.2.2 Proof of Theorem 4.3

Proof of Theorem 4.3: By Lemmas 8.9 and 8.10, for each δ and n, $T_{i\delta}^n$ is formed by the union of L_n antichains, that is, we can write $T_{i\delta}^n = \cup_{m=1}^{L_n} T_{i\delta}^{n,m}$, where $T_{i\delta}^{n,m}$ is an antichain. By defining $T_{i\delta}^{n,m} = \emptyset$ for $m > L_n$, we can consider $T_{i\delta}^{n,m}$ well defined for all $m \in \mathbb{N}$ and write $T_{i\delta}^n = \cup_{m \in \mathbb{N}} T_{i\delta}^{n,m}$.

Now, let E denote the set of types $t_i \in T_i$ which play strictly mixed strategies as a best reply. This set is included in $\cup_{n \in \mathbb{N}} T_i^n$, where T_i^n denotes the set $\{t_i \in T_i : diam\,(BR_i(t_i)) > \frac{1}{n}\}$. Thus, to show that $\tau_i(E) = 0$ it is enough to argue that $\tau_i(T_i^n) = 0$ for every $n \in \mathbb{N}$. Actually, it is enough to prove that $\tau_i(T_i^{n,m}) = 0$, where $T_i^{n,m} \equiv \cup_{\delta \in \Delta_i} T_{i\delta}^{n,m}$.

To see that $\tau_i(T_i^{n,m}) = 0$, assume otherwise; that is, there exist n, m such that $\tau_i(T_i^{n,m}) > 0$. By $(*)$, there exist $\delta \in \Delta_i$ and $t_i, t_i' \in T_i^{n,m} \cap \widehat{\delta}_i^{-1}(\delta) = T_{i\delta}^{n,m}$ such that $t_i < t_i'$, but this is an absurd, because $T_{i\delta}^{n,m}$ is an antichain. ∎

[54] Of course $x, y \leqslant z, w$ is an abbreviation for four inequalities.

8.3 Proofs for Section 5

8.3.1 Tie-breaking rule

Hereafter, let $b_i^{1\wedge 2}$ be an abbreviation for $b_i^1 \wedge b_i^2$ and $b_i^{1\vee 2}$, for $b_i^1 \vee b_i^2$.

Proof of Proposition 5.4: By the first part of Assumption 5.2 , $h_i^*(b_i^1, b_{-i}) \geqslant h_i^*(b_i^{1\wedge 2}, b_{-i})$ and $h_i^*(b_i^2, b_{-i}) \leqslant h_i^*(b_i^{1\vee 2}, b_{-i})$. For a contradiction, suppose that $h_i^*(b_i^1, b_{-i}) > h_i^*(b_i^{1\wedge 2}, b_{-i})$. Let $h - 1 = h_i^*(b_i^{1\wedge 2}, b_{-i})$. By Assumption 5.2 , this implies that either (i) $b_{ih}^{1\wedge 2} \neq s_{ih}$ or that (ii) $b_{ih}^1 \neq s_{ih}$. We need the following two observations. First, since $h_i^*(b_i^{1\wedge 2}, b_{-i}) < h$, then $b_{ih}^{1\wedge 2} \leqslant s_{ih}$. Also, since $h_i^*(b_i^1, b_{-i}) \geqslant h$, we have $b_{ih}^1 \geqslant s_{ih}$. Therefore, $b_{ih}^1 \geqslant s_{ih} \geqslant b_{ih}^{1\wedge 2}$. This implies that in case (i) we must have $b_{ih}^{1\wedge 2} < s_{ih}$ and in case (ii), we must have $b_{ih}^1 > s_{ih}$. In any case, $b_{ih}^1 > b_{ih}^{1\wedge 2}$, which implies that $b_{ih}^{1\wedge 2} = b_{ih}^2$. Since $h_i^*(b_i^{1\wedge 2}, b_{-i}) < h \leqslant h_i^*(b_i^1, b_{-i}) \leqslant h_i^*(b_i^2, b_{-i})$, and $b_{ih}^{1\wedge 2} = b_{ih}^2 \leqslant s_{ih}$, another application of Assumption 5.2 leads to $b_{ih}^2 < s_{ih}$. However, this would imply $h_i^*(b_i^2, b_{-i}) < h$, a contradiction. The proof that $h_i^*(b_i^2, b_{-i}) = h_i^*(b_i^{1\vee 2}, b_{-i})$ is analogous. ∎

8.3.2 Payment

Before we establish the proof of (17), it will be useful to introduce some notation. For this, fix (e, b) and let $m = h^*(e, b)$. If $m_i < e_i$, player i has sold $e_i - m_i$ units in the auction, while she has bought $m_i - e_i$ if $m_i > e_i$. No negotiation is made by i if $m_i = e_i$. Define $K \equiv \sum_{i=0}^N e_i$ and let $s_i = (s_{i,1}, s_{i,2}, ..., s_{i,K})$ be the profile of the K-highest bids made by players $j \neq i$, such that $s_{i,1} \leq s_{i,2} \leq ... \leq s_{i,K}$. In other words, s_i denotes the (inverse) residual supply curve facing bidder i; $s_{i,K}$ is the highest of the bids by players $j \neq i$, $s_{i,K-1}$ is the second highest and so on. Thus, for getting (for sure) at least one unit, bidder i's highest bid must be above $s_{i,1}$, that is, $b_{i,1} > s_{i,1}$. For bidder i winning at least two units for sure, it is necessary $b_{i,2} > s_{i,2}$ and so on. Figure 1 illustrates this.

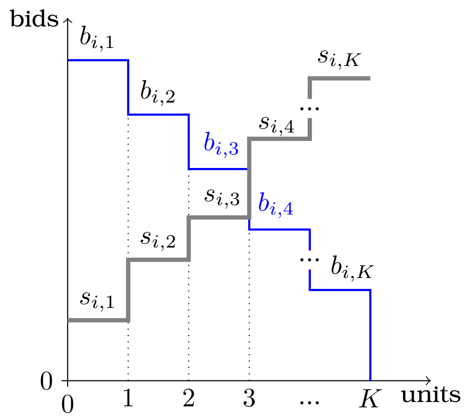

Figure 1: Bid (b_i) and supply (s_i) curves for bidder i. In the situation displayed, bidder i receives three units, because $b_{i,3} > s_{i,3}$ but $b_{i,4} < s_{i,4}$.

Given $b \in A$ and $i \in I$, let $h = \overline{h}_i(b) \equiv \max\{j : b_{i,j} \geqslant s_{i,j}\}$. Define $\underline{p}(b) \equiv \max\{s_{i,h}, b_{i,h+1}\}$ and $\overline{p}(b) \equiv \min\{b_{i,h}, s_{i,h+1}\}$. These definitions do not depend on i. Indeed, first notice that $\underline{p}(b) \leqslant \overline{p}(b)$. If there is a competitive tie, that is, $b_{ih} = s_{ih}$, then $\underline{p}(b) = \overline{p}(b)$ and both are equal to the tying bid (thus, they do not depend on i). Consider now the case with non-competitive ties, that is, $b_{ih} > s_{ih}$ and $b_{i,h+1} < s_{i,h+1}$. In this case, $\underline{p}(b)$ is the highest losing bid and $\overline{p}(b)$ is the lowest winning bid and, as such, both $\underline{p}(b)$ and $\overline{p}(b)$ do not depend on i. Finally, note that the definition would not have changed if we have used $h = h_i^*(b)$ instead of $\overline{h}_i(b)$, because these two quantities are different only when there is a competitive tie and, in this case, even if $h = h_i^*(b) < \min\{j : b_{i,j} = s_{i,j}\}$, that is, i does not receive any object in the tie, we still have $b_{i,h+1} = s_{i,h+1}$, which is sufficient to imply that both $\underline{p}(b)$ and $\overline{p}(b)$ are equal to the competitive tie.

Lemma 8.11 *Fix b_i^1, b_i^2, b_{-i} and let $\underline{p}^k = \underline{p}(b_i^k, b_{-i})$ and $\overline{p}^k = \overline{p}(b_i^k, b_{-i})$, for $k = 1, 2, 1 \wedge 2$ and $1 \vee 2$. Without loss, assume that $h^1 = h_i^*(b_i^1, b_{-i}) \leqslant h^2 = h_i^*(b_i^2, b_{-i})$. We have the following: if $h^1 < h^2$, then*

$$\underline{p}^{1\vee 2} = \underline{p}^2; \underline{p}^{1\wedge 2} = \underline{p}^1; \overline{p}^{1\vee 2} = \overline{p}^2;\ \textit{and}\ \overline{p}^{1\wedge 2} = \overline{p}^1; \tag{30}$$

and if $h^1 = h^2$, $b_{ih}^{1\vee 2} = b_{ih}^k$ and $b_{ih}^{1\wedge 2} = b_{ih}^{k'}$ for $(k, k') = (1, 2)$ or $(2, 1)$, then $\overline{p}^{1\vee 2} = \overline{p}^k$; and $\overline{p}^{1\wedge 2} = \overline{p}^{k'}$; and a similar condition holds for $\underline{p}$.

Proof. By Proposition 5.4, $h_i^*(b_i^{1\wedge 2}, b_{-i}) = h^1$ and $h_i^*(b_i^{1\vee 2}, b_{-i}) = h^2$.

Let's consider first the case that $h^1 < h^2$. This means that $b_{i,h^2}^{1\vee 2} = b_{i,h^2}^2$, otherwise, $b_{i,h^2}^2 < b_{i,h^2}^{1\vee 2} = b_{i,h^2}^1$, which would imply $b_{i,h^2}^1 \geqslant s_{i,h^2}$. In this case, Assumption 5.2 would imply $h^1 \geqslant h^2$, a contradiction. Therefore, $\overline{p}^{1\vee 2} = \overline{p}^2$. Next, we want to show that $\overline{p}^{1\wedge 2} = \min\{b_{i,h^1}^{1\wedge 2}, s_{i,h^1+1}\} = \min\{b_{i,h^1}^1, s_{i,h^1+1}\} = \overline{p}^1$.

Suppose otherwise, that is, $\min\{b^{1\wedge 2}_{i,h^1}, s_{i,h^1+1}\} < \min\{b^1_{i,h^1}, s_{i,h^1+1}\}$. Then, $b^{1\wedge 2}_{i,h^1} < b^1_{i,h^1}$ and $b^{1\wedge 2}_{i,h^1} = b^2_{i,h^1} < s_{i,h^1+1}$. But this implies $s_{i,h^2} \geqslant s_{i,h^1+1} > b^2_{i,h^1} \geqslant b^2_{i,h^2}$, which contradicts $h^2 = h^*_i(b^2_i, b_{-i})$. This shows that $\overline{p}^{1\wedge 2} = \overline{p}^1$.

Analogously, suppose that $\underline{p}^{1\wedge 2} = \max\{b^{1\wedge 2}_{i,h^1+1}, s_{i,h^1}\} < \max\{b^1_{i,h^1+1}, s_{i,h^1}\} = \underline{p}^1$. This implies that $b^2_{i,h^1+1} = b^{1\wedge 2}_{i,h^1+1} < b^1_{i,h^1+1} \leqslant s_{i,h^1+1}$, which contradicts $h^2 > h^1 \Rightarrow b^2_{i,h^1+1} \geqslant b^2_{i,h^2} \geqslant s_{i,h^2} \geqslant s_{i,h^1+1}$. Thus, $\underline{p}^{1\wedge 2} = \underline{p}^1$.

Now, assume $\underline{p}^{1\vee 2} = \max\{b^{1\vee 2}_{i,h^2+1}, s_{i,h^2}\} > \max\{b^2_{i,h^2+1}, s_{i,h^2}\} = \underline{p}^2$. This implies that $b^{1\vee 2}_{i,h^2+1} > b^2_{i,h^2+1}$ and $b^1_{i,h^2+1} = b^{1\vee 2}_{i,h^2+1} > s_{i,h^2}$, but this implies $b^1_{i,h^1+1} \geqslant b^{1\vee 2}_{i,h^2+1} = b^1_{i,h^2+1} > s_{i,h^2} \geqslant s_{i,h^1+1}$, contradicting the assumption that b^1_i receives only h^1 and not h^1+1. Thus, $\underline{p}^{1\vee 2} = \underline{p}^2$.

Now, assume that $h^1 = h^2 = h$. Let $b^{1\vee 2}_{ih} = b^k_{ih}$, $b^{1\wedge 2}_{ih} = b^{k'}_{ih}$ for $(k,k') = (1,2)$ or $(k,k') = (2,1)$. Thus, $\overline{p}^{1\vee 2} = \min\{b^{1\vee 2}_{ih}, s_{i,h+1}\} = \min\{b^k_{ih}, s_{i,h+1}\} = \overline{p}^k$ and $\overline{p}^{1\wedge 2} = \min\{b^{1\wedge 2}_{ih}, s_{i,h+1}\} = \min\{b^{k'}_{ih}, s_{i,h+1}\} = \overline{p}^{k'}$. Analogously, let $b^{1\vee 2}_{i,h+1} = b^k_{i,h+1}$, $b^{1\wedge 2}_{i,h+1} = b^{k'}_{i,h+1}$.[55] Thus, $\underline{p}^{1\vee 2} = \max\{b^{1\vee 2}_{i,h+1}, s_{i,h}\} = \min\{b^k_{i,h+1}, s_{i,h}\} = \underline{p}^k$ and $\underline{p}^{1\wedge 2} = \max\{b^{1\wedge 2}_{i,h+1}, s_{i,h}\} = \max\{b^{k'}_{i,h+1}, s_{i,h}\} = \underline{p}^{k'}$. ∎

Remark 8.12 *As the statement of this lemma should suggest, (30) is not necessarily true when $h^1 = h^2$. To see this, consider the following example. There are two objects, the two highest bids by opponents are $(6,2)$, $b^1_i = (7,3)$ and $b^2_i = (5,4)$. Then $h^1 = h^2 = 1$, $b^{1\wedge 2}_i = (5,3)$, $b^{1\vee 2}_i = (7,4)$, $\overline{p}^{1\wedge 2} = \overline{p}^2 = 5$; $\overline{p}^{1\vee 2} = \overline{p}^1 = 6$, $\underline{p}^{1\wedge 2} = \underline{p}^1 = 3$ and $\underline{p}^{1\vee 2} = \underline{p}^2 = 4$, which shows that (30) is not true, even relabeling the bids. Note also that this example does not depend on the specification of the tie-breaking rule, because there are no ties.*

Corollary 8.13 *Let Assumption 5.1 hold. Then,*

$$p_i(h, e, b^1_i, b_{-i}) - p_i(h, e, b^{1\wedge 2}_i, b_{-i}) = p_i(h, e, b^{1\vee 2}_i, b_{-i}) - p_i(h, e, b^2_i, b_{-i}).$$

Proof. Using Lemma 8.11, a straightforward inspection of (15) leads to the conclusion. ∎

8.3.3 Proof of Theorem 5.1

The proof of Theorem 5.1 requires two lemmas. The first one generalizes an argument first given by McAdams (2003, p. 1210). For the discussion below, let u_i be given by (13).

Lemma 8.14 *Under assumptions 5.1 and 5.2, u_i is modular.*

Proof. Fix b_{-i} and let $h^1, h^2, h^{1\wedge 2}$ and $h^{1\vee 2}$ be the final allocation given by b^1_i, b^2_i, $b^{1\wedge 2}_i$ and $b^{1\vee 2}_i$. Without loss of generality, assume $h^1 \leqslant h^2$. By Proposition 5.4,

[55] These k and k' could be different from the first part.

$h = h^1 = h^{1\wedge 2}$ and $h' = h^2 = h^{1\vee 2}$. For simplicity, define $p^1 \equiv p_i(h, e, \cdot, b_i^1)$; $p^{1\wedge 2} \equiv p_i(h, e, \cdot, b_i^{1\wedge 2})$; $p^2 \equiv p_i(h', e, \cdot, b_i^2)$ and $p^{1\vee 2} \equiv p_i(h', e, \cdot, b_i^{1\vee 2})$.

We want to show that

$$u_i(t_i, b_i^1, \cdot) - u_i(t_i, b_i^{1\wedge 2}, \cdot) \quad = \quad u_i(t_i, b_i^{1\vee 2}, \cdot) - u_i(t_i, b_i^2 . \cdot), \tag{31}$$

that is,

$$\begin{aligned} \left[\sum_{j=1}^{h} v_{ij} - p^1\right] - \left[\sum_{j=1}^{h} v_{ij} - p^{1\wedge 2}\right] \quad &= \quad \left[\sum_{j=1}^{h'} v_{ij} - p^{1\vee 2}\right] - \left[\sum_{j=1}^{h'} v_{ij} - p^2\right] \\ \iff p^{1\wedge 2} - p^1 \quad &= \quad p^2 - p^{1\vee 2}, \end{aligned}$$

but the last expression is true by Corollary 8.13. ∎

Lemma 8.15 *(i) u_i has nondecreasing differences.*

(ii) If $h^1 = h_i^*(b_i^1, b_{-i}) < h_i^*(b_i^2, b_{-i}) = h^2$, then the inequality defining nondecreasing differences is actually strict.

Proof. Let $t_i^1 < t_i^2$ and assume that $b_i^1 <_i b_i^2$ and let $h = h_i^*(b_i^1, b_{-i}) = h_i^*(b_i^2, b_{-i})$. For simplicity, define $p^1 \equiv p_i(h, e^1, \cdot, b_i^1)$ and $p^2 \equiv p_i(h, e^2, \cdot, b_i^2)$. Then,

$$\begin{aligned} &\left(\sum_{j=1}^{h} v_{ij}^1 - p^2\right) - \left(\sum_{j=1}^{h} v_{ij}^1 - p^1\right) \\ &= \left(\sum_{j=1}^{h} v_{ij}^2 - p^2\right) - \left(\sum_{j=1}^{h} v_{ij}^2 - p^1\right). \end{aligned}$$

Given the second claim, this will be enough for the first part.

Now, let $h^1 = h_i^*(b_i^1, b_{-i}) < h_i^*(b_i^2, b_{-i}) = h^2$. For simplicity, define $p^1 \equiv p_i(h^1, e^1, \cdot, b_i^1)$ and $p^2 \equiv p_i(h^2, e^2, \cdot, b_i^2)$. We want to show that

$$\begin{aligned} &\left(\sum_{j=1}^{h^2} v_{ij}^1 - p^2\right) - \left(\sum_{j=1}^{h^1} v_{ij}^1 - p^1\right) \\ &< \left(\sum_{j=1}^{h^2} v_{ij}^2 - p^2\right) - \left(\sum_{j=1}^{h^1} v_{ij}^2 - p^1\right), \end{aligned}$$

which is equivalent to:

$$\sum_{j=h^1+1}^{h^2} v_{ij}^1 \quad < \quad \sum_{j=h^1+1}^{h^2} v_{ij}^2.$$

Since $t_i^1 < t_i^2$, the last inequality is obviously true by (12). ∎

Proof of Theorem 5.1: Define a modified n-auction exactly as the original, except for two differences:

- with probability $1/n$, there is a player playing uniformly in all bids; and
- the tie-breaking is modified so that it satisfies Assumption 5.2.

By JS' Theorems 6 and 9, each n-modified auction has an equilibrium in undominated* strategies with a zero probability of competitive ties, which is an equilibrium under any omniscient and effectively trade-maximizing tie-breaking rule, including standard tie-breaking rule. We want to argue that this equilibrium is actually in pure strategies.

Since the tie-breaking rule of the modified auction satisfies Assumption 5.2, by Lemma 8.14, u_i is modular, hence quasi-supermodular. By Lemma 8.15, u_i has non-decreasing differences. Moreover, since in the n-modified auction there is a player playing uniformly in all bids, then $h_i^*(b_i^1, b_{-i}) < h_i^*(b_i^2, b_{-i})$ occurs with positive probability for any pair of bids b_i^1 and b_i^2 satisfying $b_i^1 < b_i^2$. Again by Lemma 8.15, the inequality defining non-decreasing differences in u_i is actually strict with positive probability. Therefore, the assumptions of Lemma 8.6 are satisfied, which implies that $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ has the monotonicity property in $T_{i\delta} \times A_i$.

Therefore, each n-auction satisfies the assumptions of Theorem 4.3 and all of its equilibria are actually in pure strategies that are monotonic when conditioned to each $T_{i\delta}$. Let b^n be the pure strategy equilibrium of the n-auction. By Jackson and Swinkels (2005, Theorem 9), this equilibrium is competitive tie-free, which means that it implies an allocation that does not depend on the tie-breaking rule. This means that we could erase the second bullet point in the definition of the n-modified auction and consider it with the original tie-breaking rule.[56]

An easy adaptation of Reny (2011, Lemma A.10) shows that for each i, $\{b_i^n\}$ has a subsequence that converges pointwise to some b_i, since each b_i^n is monotonic when restricted to $T_{i\delta}$. Naturally, $b = (b_i, b_{-i})$ can involve ties with positive probability. Let us define a specific tie-breaking rule for this case. For each $t \in T$, let the allocation determined by b^n be denoted by $a^n(t)$, which is defined up to a set of zero measure on T. Since the allocation is discrete, we can pass to subsequences, if necessary, and assume that $a^n(t)$ converge to some $a(t)$, for almost all t. Fixing this sub-sub-sequence (but maintaining the notation on n, for simplicity), $a(t)$ defines an omniscient tie-breaking rule. We will show below that this a does not really matter. Therefore we can conceive of the hypothetical game in which the knowledge to implement a is given. For this a, we have by construction that $U_i(b^n) \to U_i(b)$.

Now we claim that b is free of competitive ties for every agent. Indeed, suppose that b induces a tie for player i. By Jackson and Swinkels (2005, Lemma 8), there

[56]The only impact of using the original tie-breaking rule in the n-modified auction is that, besides all pure strategies equilibria, we may also have equilibria in mixed strategies.

exists a strategy b'_i that is tie-free for player i and satisfies

$$U_i(b'_i, b_{-i}) > U_i(b_i, b_{-i}). \tag{32}$$

Since b'_i is tie-free for player i, $u_i(t, b'_i(t_i), b^n_{-i}(t_{-i})) \to u_i(t, b'_i(t_i), b_{-i}(t_{-i}))$ for almost all t_{-i}. By Lebesgue dominated convergence theorem, $U_i(b'_i, b^n_{-i}) \to U_i(b'_i, b_{-i})$ and this, together with $U_i(b^n) \to U_i(b)$ and (32) would imply that

$$U_i(b'_i, b^n_{-i}) > U_i(b^n_i, b^n_{-i})$$

for all sufficiently high n. But since the modified n-auction has payoffs U^n_i very close to the original auction payoff U_i, we would obtain, for sufficiently high n,

$$U^n_i(b'_i, b^n_{-i}) > U^n_i(b^n_i, b^n_{-i}).$$

However, this contradicts the assumption that b^n is an equilibrium of the modified auction. Therefore, the claim is established.

Now, we argue that b is actually an equilibrium of the original auction. For this, we can assume that a player has a profitable deviation b'_i so that (32) holds. In this case, we can repeat the same arguments above and again obtain a contradiction. This implies that b is actually an equilibrium of the game with the omniscient tie-breaking rule a. However, we have argued above that b is tie free for every player i. Therefore, a does not matter for the equilibrium condition and b is also an equilibrium of the original game, as we wanted to show. ∎

9 Appendix B: Grid Distributions

9.1 Formal definition and Basic Properties

Let $\lceil x \rceil$ denote the minimum integer at least as large as x, for instance, $\lceil 2.7 \rceil$=3. For each $k \in \mathbb{N}$ and $i \in I$, define the functions $\mathbb{I}^k_i : T_i \to \{1, ..., k\}^{l_i}$ by:

$$\mathbb{I}^k_i(t_i) = \left(\left\lceil k \, \frac{t_{i,1} - \underline{t}_{i,1}}{\bar{t}_{i,1} - \underline{t}_{i,1}} \right\rceil, ..., \left\lceil k \, \frac{t_{i,j} - \underline{t}_{i,j}}{\bar{t}_{i,j} - \underline{t}_{i,j}} \right\rceil ..., \left\lceil k \, \frac{t_{i,l_i} - \underline{t}_{i,l_i}}{\bar{t}_{i,l_i} - \underline{t}_{i,l_i}} \right\rceil \right)$$

and $\mathbb{I}^k : T \to \{1, ..., k\}^L$, where $L = \sum_{i=1}^N l_i$, by $\mathbb{I}^k(t) = (\mathbb{I}^k_1(t_1), ..., \mathbb{I}^k_N(t_N))$. Note that $\mathbb{I}^k_i(t_i)$ gives the interval that contains each coordinate of t_i. We already know from the introduction that grid distributions have a constant density in each of these intervals. The following is a formal definition of grid distributions in general.

Definition 9.1 (Grid distribution) *The distribution τ is a grid distribution if it is absolutely continuous with respect to the Lebesgue measure (in $\mathbb{R}^L$), with correspondent Radon-Nikodym $f : T \to \mathbb{R}_+$ satisfying the following: there exists $k \in \mathbb{N}$ such that for almost all $t, t' \in T$,*

$$\mathbb{I}^k(t) = \mathbb{I}^k(t') \implies f(t) = f(t'). \tag{33}$$

In this case, we say that $f \in \mathcal{D}^k$. The set of densities associated to grid distributions is then $\mathcal{D}^\infty \equiv \cup_{k=1}^\infty \mathcal{D}^k$.

We will abuse terminology and say that if $\mathbb{I}^k(t) = \mathbb{I}^k(t')$ then t and t' are in the same *grid-cube*. More precisely, a *grid-cube* is a set $\{t \in T : \mathbb{I}^k(t) = c\}$ for some $c \in \{1, ..., k\}^L$. Also, we can define a *grid-interval* as a set $\{t_i \in T_i : \mathbb{I}_i^k(t_i) = c\}$ for some $c \in \{1, ..., k\}^{l_i}$ and some $i \in I$.[57] In this way, we can concisely define grid distributions as those absolutely continuous distributions with density functions that are constant in grid-cubes.

Another useful way of describing the set of grid distributions is through the image of a transformation $\mathbb{T}^k : \mathcal{D} \to \mathcal{D}$, where $\mathcal{D}$ is the set of densities in T. For simplicity, let us describe this transformation in the particular case in which $T = [0,1]^2$, that is, there are two players, each with types in $[0,1]$. In this case,

$$\mathbb{T}^k(f)(x,y) \equiv k^2 \int_{\frac{p-1}{k}}^{\frac{p}{k}} \int_{\frac{m-1}{k}}^{\frac{m}{k}} f(\alpha,\beta)\, d\alpha d\beta,$$

whenever $(x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m, p \in \{1, 2, ..., k\}$. Observe that $\mathbb{T}^k(f)$ is constant over each square $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Then $\mathcal{D}^k$ be the image of $\mathcal{D}$ by $\mathbb{T}^k$, that is, $\mathcal{D}^k \equiv \mathbb{T}^k(\mathcal{D})$. Thus, $\mathbb{T}^k$ is a projection from the infinite dimensional space $\mathcal{D}$ over the finite dimensional space $\mathcal{D}^k$. Indeed, $\mathcal{D}^k$ is finite dimensional set because any density function $f \in \mathcal{D}^k$ can be described by a matrix $A = (a_{ij})_{k \times k}$, as the figure 3 below illustrates.

The transformation $\mathbb{T}^k$ is interesting because allows us to approximate any density $f \in \mathcal{D}$ by grid distributions in a convenient way. See section 6.5 below.

9.1.1 Basic Properties

Lemma 9.2 *Let $g : (a,b] \times (c,d] \to \mathbb{R}$ be continuous. Then, there is $(x,y) \in (a,b) \times (c,d)$ such that*

$$\int_c^d \int_a^b g(\alpha,\beta)\, d\alpha d\beta = (c-d)(b-a)\, g(x,y).$$

Proof. This is a trivial application of the mean value theorem of integration. Define the function $f : (c,d] \to \mathbb{R}$ by

$$f(y) = \int_a^b g(\alpha, y)\, d\alpha.$$

It is clear that f is continuous. By the mean value theorem of integration, there exists $y \in (c,d)$ such that

$$\int_c^d \int_a^b g(\alpha,\beta)\, d\alpha d\beta = \int_c^d f(\beta)\, d\beta = (c-d)\, f(y).$$

[57] Of course a *grid-interval* is also a "(hyper)cube" in multidimensional settings, but it is just an interval if each player's type is unidimensional. Also, note that a *grid-cube* will be just a square if there are just two players with unidimensional types. We adopt this terminology to simplify later references.

Fixing y, the function $x \mapsto h(x) = g(x, y)$ is continuous. Thus, there exists $x \in (a, b)$ such that

$$f(y) = \int_a^b g(\alpha, y)\, d\alpha = \int_a^b h(\alpha)\, d\alpha = (b - a)\, h(x),$$

which concludes the proof. ∎

It is clear that the above proof easily extends to arbitrary dimensions. Therefore, we have the following:

Corollary 9.3 *Given $c \in \{1, ..., k\}^L$ and $f \in \mathcal{C}$, there exists $t \in T$ such that $\mathbb{I}^k(t) = c$ and $\mathbb{T}^k(f)(t) = f(t)$.*

9.2 Approximation results

Proof of Theorem 6.6. We first prove (iii). Given $f \in \mathcal{C}$, since T is compact, f is uniformly continuous. Thus, for any $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $\|t - t'\| < \frac{2}{k}\sqrt{L}$ implies that $|f(t) - f(t')| < \epsilon$. Now, define $f^k \equiv \mathbb{T}^k(f)$. We claim that $\|f - f^k\| < \epsilon$, that is, $|f(t) - f^k(t)| < \epsilon$, $\forall t \in T$. Fix $t \in T$ and let $c \equiv \mathbb{I}^k(t)$. By Corollary 9.3, there exists t' such that $\mathbb{I}^k(t') = c$ and $f(t') = f^k(t') = f^k(t)$. Since $\mathbb{I}^k(t) = c$, then $\|t - t'\| < \frac{2}{k}\sqrt{L}$ and we have $|f^k(t) - f(t)| = |f(t') - f(t)| < \epsilon$. This establishes (iii).[58]

For (ii), recall that continuous functions are dense in L^p (see, for instance, Aliprantis and Border (1999, Theorem 12.9)). Given $f \in L^p$ and $\epsilon > 0$, let $g \in \mathcal{C}$ be such that $\|f - g\| < \frac{\epsilon}{2}$ and pick k such that $\|\mathbb{T}^k(g) - g\| < \frac{\epsilon}{2}$. Then, $\|f - \mathbb{T}^k(g)\| < \epsilon$.

Now (i) is immediate, since $\mathbb{T}^k(f) \to f$ in the sup-norm implies that it also converges pointwise. It remains to show that the set of grid distributions is dense in Δ. Given $\tau \in \Delta$, let $f \in \mathcal{D}$ be (a version of) its Radon-Nikodym derivative with respect to the Lebesgue measure λ on T. Let $f^k \equiv \mathbb{T}^k(f)$ and τ^k the distribution generated by $f^k \in \mathcal{D}^k$. Then, for any continuous function $g : T \to \mathbb{R}$, $\int g d\tau^k = \int g f^k d\lambda \to \int g f d\lambda = \int g d\tau$, by the Lebesgue Dominated Convergence Theorem (it is not difficult to see that the sequence $g f^k$ is bounded). Since g is an arbitrary continuous function, $\tau^k \to \tau$ in the weak* topology, which establishes the first claim in the theorem. ∎

As observed in footnote 58, we also have the following:

Corollary 9.4 *If $f \in \mathcal{C}$, $\mathbb{T}^k(f) \to f$ in the sup-norm.*

Proof of Proposition 6.7. For notational simplicity, we will make the proof for the case in which $T = [0, 1]^2$. The same argument can be easily generalized. We want to prove that if $x > x'$ and $y > y'$ then

$$\mathbb{T}^k(f)(x, y)\, \mathbb{T}^k(f)(x', y') \geq \mathbb{T}^k(f)(x, y')\, \mathbb{T}^k(f)(x', y).$$

[58] This also shows that $\mathbb{T}^k(f) \to f$ in the sup-norm.

Let $x \in \left(\frac{i-1}{k}, \frac{i}{k}\right]$, $x' \in \left(\frac{j-1}{k}, \frac{j}{k}\right]$, $y \in \left(\frac{q-1}{k}, \frac{q}{k}\right]$ and $y' \in \left(\frac{p-1}{k}, \frac{p}{k}\right]$, where $i > j$ and $q > p$. Thus, the above inequality is equivalent to:

$$\begin{aligned} & \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} f\left(\alpha + \frac{i-1}{k}, \beta + \frac{q-1}{k}\right) d\alpha d\beta \cdot \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} f\left(z + \frac{j-1}{k}, w + \frac{p-1}{k}\right) dz dw \\ \geq & \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} f\left(\alpha + \frac{i-1}{k}, w + \frac{p-1}{k}\right) d\alpha dw \cdot \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} f\left(z + \frac{j-1}{k}, \beta + \frac{q-1}{k}\right) dz d\beta. \end{aligned}$$

This can be rewritten as:

$$\begin{aligned} & \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} \int_0^{\frac{1}{k}} \left[f\left(\alpha + \frac{i-1}{k}, \beta + \frac{q-1}{k}\right) f\left(z + \frac{j-1}{k}, w + \frac{p-1}{k}\right) \right. \\ & \left. - f\left(\alpha + \frac{i-1}{k}, w + \frac{p-1}{k}\right) f\left(z + \frac{j-1}{k}, \beta + \frac{q-1}{k}\right) \right] d\alpha d\beta dz dw \\ \geq & \ 0. \end{aligned}$$

Now it is easy to see that affiliation implies that the integrand is non-negative for all $\alpha, \beta, z, w \in \left(0, \frac{1}{k}\right)$.

For the converse, suppose that $\mathbb{T}^k(f) \in \mathcal{A}\ \forall k \in \mathbb{N}$ but $f \notin \mathcal{A}$. This means that there exist x, x', y and y' such that $x > x'$, $y > y'$ and

$$f(x, y) f(x', y') < f(x, y') f(x', y).$$

Since $\mathbb{T}^k(f) \to f$ in the sup-norm (Corollary 9.4), there exists $k \in \mathbb{N}$ such that

$$\mathbb{T}^k(f)(x, y) \mathbb{T}^k(f)(x', y') < \mathbb{T}^k(f)(x, y') \mathbb{T}^k(f)(x', y),$$

which contradicts $\mathbb{T}^k(f) \in \mathcal{A}$. ∎

Proof of Proposition 6.8 By the Lebesgue Dominated Convergence Theorem,

$$U_i(s^{k_n}) = \int u_i(t, s^{k_n}(t)) f^{k_n}(t) \lambda(dt) \to \int u_i(t, s(t)) f(t) \lambda(dt) = U_i(s).$$

Thus, $U_i(s_i, s_{-i}) \geqslant U_i(s_i', s_{-i}) = \lim_{n\to\infty} U_i(s_i', s_{-i}^{k_n})$ for all $s_i' \in \mathcal{F}_i$. ∎

Proof of Proposition 6.9 By the previous results, there exists k_ε such that for all $k \geqslant k_\varepsilon$,

$$\begin{aligned} \int u_i(t, s(t)) f^k(t) \lambda(dt) & \geqslant \int u_i(t, s(t)) f(t) \lambda(dt) - \frac{\varepsilon}{2} \\ & \geqslant \int u_i(t, s_i'(t_i), s_{-i}(t_{-i})) f(t) \lambda(dt) - \frac{\varepsilon}{2} \\ & \geqslant \int u_i(t, s_i'(t), s_{-i}(t_{-i}) f^k(t) \lambda(dt) - \varepsilon, \end{aligned}$$

where the second inequality comes from the fact that s is an equilibrium for f. ∎

9.3 Steps in the proof of Theorem 6.2

We assume that $f \in \mathcal{D}^k$ is symmetric and can be described by the matrix $(a_{ij})_{k\times k}$ as follows:

$$f(y,x) = a_{mp} \text{ if } (y,x) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right], \tag{34}$$

for $m,p \in \{1,2,...,k\}$. The definition of f at the zero measure set of points $\{(y,x) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

For the description of the steps below, assume that $x \in \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and $z \in \left(\frac{m-1}{k}, \frac{m}{k}\right]$, for $m,p \in \{1,...,k\}$.

1. We obtain the expressions:

$$\begin{aligned} f(z|x) &= \frac{ka_{mp}}{\sum_{i=1}^{k} a_{ip}}; \\ F(z|x) &= \int_0^z f(\alpha|x)\,d\alpha = \frac{\sum_{i=1}^{m-1} a_{ip} + a_{mp}(kz-m+1)}{\sum_{i=1}^{k} a_{ip}} \end{aligned}$$

and

$$\frac{f(z|x)}{F(z|x)} = \frac{ka_{mp}}{\sum_{i=1}^{m-1} a_{ip} + a_{mp}(kz-m+1)}. \tag{35}$$

2. Using (35), we integrate $b(z) = z - \int_0^z \exp\left[-\int_u^z \frac{g(s|s)}{G(s|s)}ds\right]du$ to obtain:

$$b\left(\frac{m-1+\zeta}{k}\right) = \frac{m-1+\zeta}{k} - \frac{(r_m+\zeta)}{2k} + \frac{\left(r_m^2 - D_m\right)}{2k(r_m+\zeta)}. \tag{36}$$

where $\zeta = kz - m + 1$ and, for $m \geqslant 2$,

$$r_m \equiv \frac{\sum_{i=1}^{m-1} a_{im}}{a_{mm}}, \tag{37}$$

and

$$D_m \equiv \sum_{j=1}^{m-1}\left[\prod_{l=j}^{m-1}\left(\frac{r_{l+1}}{1+r_l}\right)\right]\cdot\left[(1+r_j)^2 - r_j^2\right]. \tag{38}$$

3. Define $\Delta(x,z) \equiv \Pi(x,b(z)) - \Pi(x,b(x))$, where $\Pi(x,b(z))$ is given by $\Pi(x,b(z)) = [x - b(z)]\,F(z|x)$. Note that $b(\cdot)$ is SMPSE iff $\Delta(x,z) \leqslant 0$ for all $(x,z) \in [0,1]^2$. Using $\chi \equiv kx - p + 1$ or $x = \frac{p-1+\chi}{k}$ and $\zeta \equiv kz - m + 1$ or $z = \frac{m-1+\zeta}{k}$, denote $2k\sum_{i=1}^{k} a_{ip}\Delta(x,z)$ by $\Delta_{pm}(\chi,\zeta)$ and obtain:

$$\begin{aligned} \Delta_{pm}(\chi,\zeta) \equiv & \left[2(\chi - \zeta + p - m) + \frac{\zeta^2 + 2\zeta r_m + D_m}{r_m + \zeta}\right] \\ & \cdot\left(\sum_{i=1}^{m-1} a_{ip} + a_{mp}\zeta\right) - \left(\chi^2 + 2\chi r_p + D_p\right)a_{pp}. \end{aligned} \tag{39}$$

4. We show that (39) is a quadratic function of $\chi \in (0,1)$ that is: non-positive if $m = p$; decreasing in $(0,1)$ if $m < p$ and increasing in $(0,1)$ if $m > p$. Thus, $b(\cdot)$ is SMPSE iff for all $m, p \in \{1, ..., k\}, m \neq p$:

$$\begin{cases} \Delta_{pm}(0,\zeta) \leqslant 0, \forall \zeta \in [0,1], \text{ if } m < p; \\ \Delta_{pm}(1,\zeta) \leqslant 0, \forall \zeta \in [0,1], \text{ if } m > p. \end{cases} \tag{40}$$

Since $r_m + \zeta > 0$, the signal of $\Delta_{pm}(\chi,\zeta)$ is the same as the signal of

$$\tilde{\Delta}_{pm}(\chi,\zeta) \equiv (r_m + \zeta)\,\Delta_{pm}(\chi,\zeta). \tag{41}$$

From (40), $b(\cdot)$ is SMPSE iff, for all $m < p$ we have $\tilde{\Delta}_{pm}(0,\zeta) \leqslant 0$ and for $m > p$, $\tilde{\Delta}_{pm}(1,\zeta) \leqslant 0$.

5. From (39), it is easy to see that both $\tilde{\Delta}_{pm}(0,\zeta)$ and $\tilde{\Delta}_{pm}(1,\zeta)$ are polynomials of third degree in ζ which do not depend on χ. We carry χ below only to consider both cases in just one expression. We then show that $\partial_\zeta \tilde{\Delta}_{pm}(\chi,\zeta) = 0$ can be written as

$$c_2\zeta^2 + c_1\zeta + c_0 = 0, \tag{42}$$

where

$$\begin{aligned} c_2 &= -3a_{mp}; \\ c_1 &= 4a_{mp}(\chi + p - m) - 2\sum_{i=1}^{m-1} a_{ip}; \\ c_0 &= (\chi + p - m)\left[2a_{mp}r_m + 2\sum_{i=1}^{m-1} a_{ip}\right] - \left(\chi^2 + 2\chi r_p + D_p\right)a_{pp} + a_{mp}D_m. \end{aligned}$$

Let $\zeta_{mp}(\chi)$ denote the solution(s) to the quadratic equation (42). Now, condition (40) requires us to test:

- if $m < p$, whether $\zeta_{mp}(0) \in (0,1)$ and if this happens, test whether $\tilde{\Delta}_{pm}\left(0, \zeta_{mp}(0)\right) \leqslant 0$;
- if $m > p$, whether whether $\zeta_{mp}(1) \in (0,1)$ and if this happens, test whether $\tilde{\Delta}_{pm}\left(1, \zeta_{mp}(1)\right) \leqslant 0$.

This concludes the method.

The above definitions allow us to a more explicit statement of Theorem 6.2.

Theorem 9.5 *Suppose that there are two risk neutral players and $f \in \mathcal{D}^k$ is described by a matrix $(a_{ij})_{k\times k}$ as in (34). Let $\tilde{\Delta}_{pm}(\chi,\zeta)$ be defined by (41). Then f has a SMPSE if and only if all of the following inequalities are satisfied:*[59]

[59]The conditions in Table 1 already incorporate some further simplifications for the case $m = 1$, not explained above.

Case	Conditions to verify
$2 \leqslant m < p \leqslant k$	$\tilde{\Delta}_{pm}(0,0) \leqslant 0$
$2 \leqslant m < p \leqslant k$	$\tilde{\Delta}_{pm}\left(0, \zeta_{mp}(0)\right) \leqslant 0$
$1 \leqslant p < m \leqslant k$	$\tilde{\Delta}_{pm}(1,1) \leqslant 0;$
$1 \leqslant p < m \leqslant k$	$\tilde{\Delta}_{pm}\left(1, \zeta_{mp}(1)\right) \leqslant 0$

Table 1 - Necessary and sufficient conditions for equilibrium existence.

It is easy to see that in each square, we need to check less than six points. Thus, the number of inequalities to test is less than $6 \cdot \frac{k(k+1)}{2} = 3k^2 + 3k$.

9.4 Proof of Theorem 6.5.

The dominant strategy for each bidder in the second price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$
\begin{aligned}
P^2 &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} y f(y|x)\, dy \cdot f(x)\, dx = \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} [y - b(y)] f(y|x)\, dy \cdot f(x)\, dx + \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y) f(y|x)\, dy \cdot f(x)\, dx,
\end{aligned}
$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx$, from the first order condition: $b^{'}(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$
\begin{aligned}
& \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y) f(y|x)\, dy \cdot f(x)\, dx \\
= & \int_{[\underline{t},\bar{t}]} \left[b(x) F(x|x) - \int_{[\underline{t},x]} b^{'}(y) F(y|x)\, dy \right] \cdot f(x)\, dx \\
= & \int_{[\underline{t},\bar{t}]} b(x) F(x|x) \cdot f(x)\, dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

In the last line, the first integral is just the expected payment for the first price auction, P^1. Thus, we have

$$
\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$
\begin{aligned}
b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\
&= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

This is the desired expression. ∎

References

ALIPRANTIS, C., AND K. C. BORDER (1999): *Infinite dimensional analysis*. Springer New York.

ARAUJO, A., AND L. I. DE CASTRO (2009): “Pure strategy equilibria of single and double auctions with interdependent values,” *Games and Economic Behavior*, 65(1), 25 – 48.

ATHEY, S. (2001): “Single Crossing Properties and the Existence of Prue Strategy Equilibria in Games of Incomplete Information,” *Econometrica*, 69(4), 861–99.

BALDER, E. (1988): "Generalized equilibrium results for games with incomplete information," *Mathematics of Operations Research*, 13(2), 265–276.

CRÉMER, J., AND R. P. MCLEAN (1985): "Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent," *Econometrica*, 53(2), pp. 345–361.

DE CASTRO, L. (2007): "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions," Discussion paper, Working paper, Universidad Carlos III de Madrid.

——— (2008): "Grid Distribution to Study Single Obejct Auctions," Discussion paper, University of Illinois at Urbana-Champaign.

——— (2011): "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions," Discussion paper, Northwestern University.

DE CASTRO, L. I., AND D. H. KARNEY (2011): "Equilibria Existence and Characterization in Auctions: Achievements and Open Questions," *Journal of Economic Surveys*, pp. no–no.

DE CASTRO, L. I., AND H. J. PAARSCH (2010): "Testing Affiliation in Private-Values Models of First-Price Auctions Using Grid Distributions," *The Annals of Applied Statistics*, 4(4), 2073–2098.

DILWORTH, R. (1950): "A decomposition theorem for partially ordered sets," *Annals of Mathematics*, 51(1), 161–166.

FANG, H., AND S. MORRIS (2006): "Multidimensional private value auctions," *Journal of Economic Theory*, 126(1), 1 – 30.

HARSANYI, J. C. (1967-8): "Games with incomplete information played by "Bayesian" players, Parts I, II, III," *Management Sci.*, 14, 159–182, 320–334, 486–502.

HEIFETZ, A., AND Z. NEEMAN (2006): "On the generic (im) possibility of full surplus extraction in mechanism design," *Econometrica*, 74(1), 213–233.

HEIFETZ, A., AND D. SAMET (1998): "Topology-Free Typology of Beliefs," *Journal of Economic Theory*, 82(2), 324 – 341.

JACKSON, M. O., L. K. SIMON, J. M. SWINKELS, AND W. R. ZAME (2002): "Communication and Equilibrium in Discontinuous Games of Incomplete Information," *Econometrica*, 70(5), 1711–1740.

JACKSON, M. O., AND J. M. SWINKELS (2005): "Existence of Equilibrium in Single and Double Private Value Auctions," *Econometrica*, 73(1), 93–139.

KHAN, M. A., K. P. RATH, AND Y. SUN (1999): "On a private information game without pure strategy equilibria," *Journal of Mathematical Economics*, 31(3), 341 – 359.

LEWIS, M. (2010): *The Big Short: Inside the Doomsday Machine*. WW Norton & Company.

LIZZERI, A., AND N. PERSICO (2000): “Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price,” *Games and Economic Behavior*, 30(1), 83 – 114.

MASKIN, E., AND J. RILEY (2000): “Equilibrium in Sealed High Bid Auctions,” *The Review of Economic Studies*, 67(3), 439–454.

MCADAMS, D. (2003): “Isotone Equilibrium in Games of Incomplete Information,” *Econometrica*, 71(4), 1191–1214.

MCADAMS, D. (2007): “On the failure of monotonicity in uniform-price auctions,” *Journal of Economic Theory*, 137(1), 729 – 732.

MERTENS, J., AND S. ZAMIR (1985): “Formulation of Bayesian analysis for games with incomplete information,” *International Journal of Game Theory*, 14(1), 1–29.

MILGROM, P., AND R. WEBER (1982): “A Theory of Auctions and Competitive Bidding,” *Econometrica*, 50(5), 1089–1122.

MONTEIRO, P., AND H. MOREIRA (2006): “First-price auctions without affiliation,” *Economics Letters*, 91(1), 1–7.

MULLER, A., AND M. SCARSINI (2006): “Stochastic Order Relations and Lattices of Probability Measures,” *SIAM Journal on Optimization*, 16(4), 1024–1043.

NEEMAN, Z. (2004): “The relevance of private information in mechanism design,” *Journal of Economic Theory*, 117(1), 55 – 77.

PERRY, M., AND P. RENY (1999): “On the Failure of the Linkage Principle in Multi-unit Auctions,” *Econometrica*, 67(4), 895–900.

RADNER, R., AND R. W. ROSENTHAL (1982): “Private Information and Pure-Strategy Equilibria,” *Mathematics of Operations Research*, 7(3), 401–409.

RENY, P. (1999): “On the existence of pure and mixed strategy Nash equilibria in discontinuous games,” *Econometrica*, 67(5), 1029–1056.

RENY, P. J. (2011): “On the Existence of Monotone Pure-Strategy Equilibria in Bayesian Games,” *Econometrica*, 79(2), 499–553.

RENY, P. J., AND S. ZAMIR (2004): “On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions,” *Econometrica*, 74(4), 1105–25.

RUSTICHINI, A. (1993): “Mixing on function spaces,” *Economic Theory*, 3(1), 183–191.

SAVANI, R., AND B. VON STENGEL (2006): “Hard-to-solve bimatrix games,” *Econometrica*, 74(2), 397–429.

TROTTER, W. (1992): *Combinatorics and partially ordered sets: Dimension theory*. Johns Hopkins Universtity Press.

VAN ZANDT, T. (2010): "Interim Bayesian Nash equilibrium on universal type spaces for supermodular games," *Journal of Economic Theory*, 145(1), 249 – 263.

VAN ZANDT, T., AND X. VIVES (2007): "Monotone equilibria in Bayesian games of strategic complementarities," *Journal of Economic Theory*, 134(1), 339 – 360.

WILSON, R. (1969): "Competitive bidding with disparate information," *Management Science*, 15, 446–448.

WILSON, R. (1977): "A bidding model of perfect competition," *The Review of Economic Studies*, 44(3), 511–518.