AMBIGUITY AVERSION
WITH THREE OR MORE OUTCOMES

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Ambiguous choice problems which involve three or more outcome values can reveal aspects of ambiguity aversion which cannot be displayed in the classic two-outcome Ellsberg urn problems, and hence are not always captured by models designed to accommodate them. This is primarily due to features of the models which have little bite in the classic examples but which impose strong restrictions in choice over more general prospects. This paper considers several such examples and examines how the standard models of ambiguity aversion perform in such cases.

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Consider the following decision problems. The *Slightly Bent Coin Problem* involves two sources of ambiguity. One source is a balanced coin which has been slightly bent. It still has some well-defined probability, in the sense that if it were to be flipped millions of times, there is some fixed value to which the proportion of heads would converge – you just don't know what that value is, and you only get to flip once. In this sense it exhibits exactly the same type of ambiguity as displayed by the event black (or yellow) in the classic Three-Color Ellsberg Urn – repeated sampling with replacement would also yield some fixed limiting proportion of black draws, but again, you don't know what that proportion is. The only difference is that since the coin is only slightly bent, you know that *its* unknown proportion is very close to one half. The other source of ambiguity in the problem is an urn containing a ball, which could be either black or white. The mechanics of the coin flip does not depend upon the contents of the urn, and the coin is flipped and the ball drawn simultaneously. The bets are based on the outcome of the flip and the color of the ball.

SLIGHTLY BENT COIN PROBLEM

	BET I				BET II	
	black	white			black	white
heads	+\$8,000	\$0	vs.	heads	\$0	\$0
tails	–\$8,000	\$0		tails	–\$8,000	+\$8,000

The next problem, the *Thermometer Problem*, involves bets on the temperature in Timbuktu at noon next May Day. The thermometer is more than just a very accurate digital thermometer – it's a perfectly accurate *analogue* thermometer, which can exactly report any value in the continuum. Divide the continuum of feasible temperatures into an extremely large number of equal-length intervals. Bet 1 yields its prize if the temperature *t* lands in the left 45% of any interval and \$0 otherwise, and Bets 2, 3 and 4 have a similar structure.

THERMOMETER PROBLEM

BET 1 (\$6,000 if t in left $\frac{45}{100}$ of any interval) vs. BET 2 (\$3,000 if t in left $\frac{90}{100}$ of any interval)

BET 3 (\$6,000 if t in left $\frac{1}{1{,}000}$ of any interval) vs. BET 4 (\$3,000 if t in left $\frac{2}{1{,}000}$ of any interval)

The final problem, the *Upper/Lower Tail Problem*, involves two urns with identical ambiguity properties: each contains exactly one red ball, along with two other balls, each of which could be either black or white. Each urn's bet involves the same prizes \$0, \$$c$ and \$100, where \$$c$ is your certainty equivalent of an objective 50:50 lottery of \$0 versus \$100. In each bet the outcome \$$c$ is ambiguous. The difference between the choices is that in Urn I this ambiguity is across the lower tail outcomes \$0 and \$$c$, whereas in Urn II it is across the upper tail outcomes \$$c$ and \$100.

UPPER/ LOWER TAIL PROBLEM

URN I				URN II		
two balls		one ball		one ball	two balls	
black	white	red	vs.	red	black	white
\$0	\$$c$	\$100		\$0	\$$c$	\$100

I. Introduction

The concept of *objective uncertainty* dates back at least to 17th Century French gamblers such as Pascal and Fermat, and mathematicians have since developed the theory of probability far beyond what is needed (or could even be applied) by economists or other decision theorists. Although humans faced situations of *subjective uncertainty* (plagues, earthquakes…) long before the invention of dice or roulette wheels, the formal recognition and specification of subjective uncertainty as a distinct concept is much more recent.[1] More recent still is the formal development of *subjective probability*, in which the theory of probability can be applied to an individual's beliefs – and hence their decisions – in subjective settings, and which has typically been posited jointly with expected utility risk preferences and termed *subjective expected utility*.[2]

While the combination of subjective probability theory with classical expected utility theory would seem to constitute the ideal framework for the analysis of choice under uncertainty, a still more recent phenomenon has caused researchers to question the empirical validity of the subjective probability hypothesis. These are the well-known thought experiments proposed by Daniel Ellsberg (1961, 1962, 2001). So-called *Ellsberg urns* present situations of objective, subjective, and *mixed objective*/*subjective* uncertainty, and most individuals' preferences for bets on such urns seem to systematically violate the existence of well-defined subjective probabilities. This feature of preferences has been termed *ambiguity aversion*. Economists and others have responded to this phenomenon by developing models – typically generalizations of subjective expected utility – designed to accommodate ambiguity aversion, and such models have been usefully applied to the analysis of economic behavior.

While they successfully model behavior in the classic Ellsberg (1961) examples, one can argue that such models may be too influenced by the specific features of those particular examples, to the extent that they may fail to capture aspects of ambiguity aversion in even slightly more general settings. In particular, all of Ellsberg's classic examples involve bets over a single pair of outcome values (Ellsberg used \$0 and \$100). Each of the major models is of course defined over multiple-outcome prospects. But if they are to have any use for economic

[1] E.g., Keynes (1921), Knight (1921).

[2] E.g., Savage (1954), Anscombe and Aumann (1963).

analysis, they must also be able to capture aspects of ambiguity aversion over such prospects.

The purpose of this paper is to consider some decision problems involving three or more outcome values, and examine how the major models of ambiguity aversion perform in such cases. Are they capable of modeling Ellsberg- (or Allais-) type behavior in these more general settings? For each decision problem, we examine the senses in which ambiguity averse (or ambiguity loving) preferences might depart from classical subjective expected utility preferences, and whether the models are capable of predicting – or even allowing for – such departures. Not all the examples involve the usual Ellsberg-type mix of objective and subjective uncertainty – some involve only purely subjective uncertainty. Thus, some of the difficulties with these models cannot be ascribed just to how they handle Ellsberg-type mixed uncertainty, but involve their deeper features.[3]

The following section reviews the classic Ellsberg urn examples and some of the models which have developed in response to them. Sections III, IV and V present examples which reveal aspects of ambiguity and ambiguity aversion which can arise in a world with three or more outcomes, and examine how the models perform in such cases. Section VI concludes.

[3] Although three-outcome examples have been discussed by Segal (1992), Diecidue and Wakker (2001), Nau (2001), Zhang (2002), Ergin and Gul (2009), Wakker (2010), even Ellsberg (1961,1962,2001), and experimentally examined by MacCrimmon and Larsson (1979,Sect.8), Tversky and Kahneman (1992), Fennema and Wakker (1996), Wu and Gonzalez (1999), Abdellaoui, Vossmann and Weber (2005), Diecidue, Wakker and Zeelenberg (2007), L'Haridon and Placido (2008,2010), Hey, Lotito and Maffioletti (2010), Baillon, L'Haridon and Placido (2011) and others, to my knowledge none of the examples presented in this paper have appeared in the literature.

II. Classic Urns and Major Models

A. Classic Ellsberg Urn Problems

In his 1961 article and 1962 PhD thesis,[4] Ellsberg presented a class of decisions problems involving both subjective and objective uncertainty, which seem to contradict the classic subjective expected utility hypothesis as axiomatized and formalized in Savage (1954). The example now known as the *Three-Color Ellsberg Paradox* involves an opaque urn containing 90 balls. Exactly 30 of these balls are known to be red, and each of the other 60 is either black or yellow, but the exact numbers are unknown, and could be anywhere from 0:60 to 60:0. A ball is to be drawn from the urn, and the decision maker is presented with two pairs of bets based on the color of the ball, as illustrated below:[5]

THREE-COLOR ELLSBERG PARADOX

	30 balls	60 balls	
	red	black	yellow
a_1	\$100	\$0	\$0
a_2	\$0	\$100	\$0
a_3	\$100	\$0	\$100
a_4	\$0	\$100	\$100

[4] Ellsberg's thesis has since been published as Ellsberg (2001).

[5] Ellsberg (1961, pp.653-656; 2001, pp.155-158). Ellsberg (2001, pp.137-142) discusses an essentially equivalent version with the payoffs \$100:\$0 replaced by \$0:–\$100.

Ellsberg conjectured, and subsequent experimenters have found,[6] that most individuals would prefer bet a_1 over bet a_2, and bet a_4 over bet a_3, which we will refer to as *Ellsberg preferences* in this choice problem. The example is termed a "paradox" since such preferences directly contradict the subjective probability hypothesis – if the individual did assign subjective probabilities to the events $\{\text{red}, \text{black}, \text{yellow}\}$, then the strict preference ranking $a_1 \succ a_2$ would reveal the strict subjective probability ranking prob(red) > prob(black), but the strict ranking $a_3 \prec a_4$ would reveal the strict ranking prob(red) < prob(black).

The widely accepted reason for these rankings is that while the bet a_1 guarantees a known probability 1/3 of winning the $100 prize, the probability of winning offered by a_2 is unknown, and could be anywhere from 0 to 2/3. Although the range [0,2/3] has 1/3 as its midpoint, and there is no reason to expect any asymmetry, individuals seem to prefer the known to the unknown probability. Similarly, bet a_4 offers a guaranteed 2/3 chance of winning, whereas the probability offered by a_3 could be anywhere from 1/3 to 1. Again, individuals prefer the known-probability bet. Ellsberg referred to bets a_2 and a_3 as involving *ambiguity*, and a preference for known-probability over ambiguous bets has come to be known as *ambiguity aversion*.

Ellsberg's article contained two additional widely-cited examples. In the left-hand example below, known as the *Two-Urn Ellsberg Paradox*,[7] Urn I contains 100 red and black balls in unknown proportions, and Urn II contains exactly 50 red and 50 black balls. Again, typical preferences are for the known-probability bets b_2 and b_4 over their unknown-probability counterparts b_1 and b_3, again

[6] See the surveys in MacCrimmon and Larsson (1979), Camerer and Weber (1992), Kelsey and Quiggin (1992), Siniscalchi (2008) and Hey, Lotito and Maffioletti (2010).

[7] Ellsberg (1961, pp.650-651,653).

contradicting the subjective probability hypothesis[8] and reflecting the same type of ambiguity aversion as in the Three-Color Paradox. In the right-hand example, known as the *Four-Color Ellsberg Paradox*,[9] the typical rankings $c_1 \succ c_2$ and $c_3 \prec c_4$ imply prob(green) > prob(black) and prob(green) < prob(black) respectively, and reveal the same type of ambiguity aversion as in the previous examples. Ellsberg observed that such examples can be viewed as providing systematic violations of Savage's *Sure-Thing Principle*.[10] As mentioned, these and related examples have received a great deal of experimental confirmation (see Note 6).

TWO-URN ELLSBERG PARADOX

	URN I		URN II	
	100 balls		50 balls	50 balls
	red	green	black	yellow
b_1	\$100	\$0		
b_2			\$100	\$0
b_3	\$0	\$100		
b_4			\$0	\$100

FOUR-COLOR ELLSBERG PARADOX

(single urn)

	100 balls		50 balls	50 balls
	red	green	black	yellow
c_1	\$100	\$100	\$0	\$0
c_2	\$100	\$0	\$100	\$0
c_3	\$0	\$100	\$0	\$100
c_4	\$0	\$0	\$100	\$100

[8] $b_1 \prec b_2$ would reveal prob(red) < prob(black), but $b_3 \prec b_4$ would reveal prob(green) < prob(yellow), violating the requirement that these probabilities satisfy prob(red) + prob (green) = prob(black) + prob (yellow) = 1.

[9] Ellsberg (1961, p.651, note 1).

[10] Savage (1954, p.23, Postulate P2). These three examples also violate Axiom P4* (*Strong Comparative Probability*) of Machina and Schmeidler (1992, p.761).

B. Major Models of Ambiguity Aversion

Ellsberg's examples have spurred the development of models which generalize and/or weaken the classic Subjective Expected Utility Model to allow for ambiguity aversion. In the finite-outcome setting, the objects of choice consist of *purely objective lotteries* $\mathbf{P} = (\ldots; x_j, p_j; \ldots)$ yielding x_j with probability p_j for some (say, monetary) outcome set $\mathcal{X} = \{x\}$, *purely subjective acts* $f(\cdot) = [x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$ yielding x_i on event E_i for some partition $\{E_1, \ldots, E_n\}$ of a subjective state space $\mathcal{S} = \{\ldots, s, \ldots\}$ or $\mathcal{S} \subseteq \mathrm{R}^{\mathrm{N}}$, and *mixed objective/subjective bets*[11] $[\mathbf{P}_1 \text{ on } E_1; \ldots; \mathbf{P}_n \text{ on } E_n]$, which are subjective bets whose "outcomes" consist of objective lotteries $\mathbf{P}_i = (\ldots; x_{ij}, p_{ij}; \ldots)$. The family of mixed objective/ subjective bets is seen to include the family of purely objective lotteries and the family of purely subjective acts.

Classical *subjective expected utility* preferences over such prospects can be represented by a preference function which takes the form

$$(1) \qquad W_{SEU}(\ldots; x_i \text{ on } E_i; \ldots) \;=\; \sum_i U(x_i) \cdot \pi(E_i)$$

over purely subjective acts, and more generally, the form

$$(2) \qquad W_{SEU}(\ldots;(\ldots; x_{ij}, p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \sum_i \Big[\sum_j U(x_{ij}) \cdot p_{ij}\Big] \cdot \pi(E_i)$$

over mixed objective/subjective bets, for some increasing cardinal *utility function* $U(\cdot)$ over outcomes and additive *subjective probability measure* $\pi(\cdot)$ over events.

[11] Such bets are also known as *Anscombe-Aumann acts* (Anscombe and Aumann (1963)).

One of the major models of ambiguity aversion over subjective or mixed objective/subjective bets is the *Rank-Dependent* (or *Choquet*) *Model* of Schmeidler (1989),[12] which takes the form

$$(3) \qquad W_{RD}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \sum_i U(x_i)\cdot\big(C(\cup_{k=1}^{i}E_k)-C(\cup_{k=1}^{i-1}E_k)\big)$$

over purely subjective acts, and more generally

$$(4) \quad W_{RD}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \sum_i\big[\sum_j U(x_{ij})\cdot p_{ij}\big]\cdot\big(C(\cup_{k=1}^{i}E_k)-C(\cup_{k=1}^{i-1}E_k)\big)$$

over mixed objective/subjective bets, for some nonadditive measure $C(\cdot)$ termed a *capacity*, and where in (3) the outcomes x_i and their corresponding events E_i are labeled so that $x_1 \succcurlyeq \ldots \succcurlyeq x_n$, and in (4) the conditional lotteries $(\ldots;x_{ij},p_{ij};\ldots)$ and their corresponding events E_i are labeled so that $\sum_j U(x_{1j})\cdot p_{1j} \succcurlyeq \ldots \succcurlyeq \sum_j U(x_{nj})\cdot p_{nj}$.[13] The intuition behind the use of a nonadditive measure is that the union of two ambiguous events (such as black and yellow in the Three-Color Urn) could well be purely objective, and it requires a nonadditive measure over events to capture this. The event $\cup_{k=1}^{i}E_k$ on which a payoff of at least x_i is received is sometimes referred to as the bet's *good-news event* for the outcome level x_i.

A second model, formalized by Gilboa and Schmeidler (1989) and termed the *Multiple Priors* (or *Maxmin Expected Utility*) *Model*, captures ambiguity aversion by means of the form

$$(5) \qquad W_{MP}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}_0}\sum_i U(x_i)\cdot\pi(E_i)$$

[12] See also Gilboa (1987), which derives from an earlier version of Schmeidler (1989).

[13] Schmeidler (1989,Theorem (pp.578-579)).

over purely subjective acts, and more generally

$$(6) \qquad W_{MP}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}_0} \sum_i \left[\sum_j U(x_{ij})\cdot p_{ij}\right]\cdot\pi(E_i)$$

for some increasing cardinal $U(\cdot)$ and some family $\mathcal{P}_0$ of subjective probability measures $\pi(\cdot)$ over events. The intuition behind this form is that an ambiguity averter evaluates each subjective or mixed bet in the most pessimistic way, given the family of measures $\mathcal{P}_0$.[14]

A more recently proposed model is the *Smooth Ambiguity Model* of Klibanoff, Marinacci and Mukerji (2005),[15] developed in part to eliminate the "kinks at certainty" properties of the Rank-Dependent and Multiple Priors forms. This model takes the form

$$(7) \qquad W_{SM}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \int_{\pi(\cdot)\in\mathcal{P}} \phi\Big(\sum_i U(x_i)\cdot\pi(E_i)\Big)\cdot d\mu\big(\pi(\cdot)\big)$$

over purely subjective acts, and more generally

$$(8) \qquad W_{SM}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \int_{\pi(\cdot)\in\mathcal{P}} \phi\Big(\sum_i \Big[\sum_j U(x_{ij})\cdot p_{ij}\Big]\cdot\pi(E_i)\Big)\cdot d\mu(\pi(\cdot))$$

for some increasing cardinal functions $U(\cdot)$ and $\phi(\cdot)$, the family $\mathcal{P}$ of *all* subjective probability measures $\pi(\cdot)$ over events, and subjective probability measure $\mu(\cdot)$ over $\mathcal{P}$. For each $\pi(\cdot)$, the expected utility of the mixed objective/subjective prospect $(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots)$ would be $\sum_i[\sum_j U(x_{ij})\cdot p_{ij}]\cdot\pi(E_i)$, and the individual is averse to the uncertainty in these expected utility levels which results

[14] Schmeidler (1986, Prop.3; 1989, pp.582-584) provides conditions under which the Multiple Priors Model contains the Rank-Dependent Model as a special case.

[15] See also the earlier analysis of Segal (1987).

from their subjective uncertainty about $\pi(\cdot)$ as represented by $\mu(\cdot)$. Risk aversion over objective uncertainty is captured by concavity of the utility function $U(\cdot)$, and ambiguity aversion captured by concavity of $\phi(\cdot)$.

The fourth major model of ambiguity aversion is the *Variational Preferences Model* of Maccheroni, Marinacci and Rustichini (2006), which takes the form

$$(9) \qquad W_{VP}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}} \Big(\sum_i U(x_i)\cdot \pi(E_i) + c\big(\pi(\cdot)\big)\Big)$$

over purely subjective acts, and more generally

$$(10) \quad W_{VP}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}} \Big(\sum_i \big[\sum_j U(x_{ij})\cdot p_{ij}\big]\cdot \pi(E_i) + c\big(\pi(\cdot)\big)\Big)$$

for some increasing cardinal function $U(\cdot)$, the family $\mathcal{P}$ of all subjective probability measures $\pi(\cdot)$ over events, and convex function $c(\cdot)$ over $\mathcal{P}$. For each $\pi(\cdot)$, the expected utility of the mixed objective/subjective bet $(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots)$, namely $\sum_i(\sum_j U(x_{ij})\cdot p_{ij})\cdot\pi(E_i)$, is supplemented by a value $c(\pi(\cdot))$ representing the individual's attitudes toward ambiguity, and the combined value then minimized over the family $\mathcal{P}$. The above authors have shown how this model includes the Multiple Priors Model and the *Multiplier Preferences Model* of Hansen and Sargent (2001) as special cases.

It is important to note how these models are applied to mixed objective/subjective prospects. In order to fully separate and represent their objective and subjective uncertainty, such prospects are expressed in the Anscombe-Aumann form $[\mathbf{P}_1 \text{ on } E_1;\ldots;\mathbf{P}_n \text{ on } E_n]$, and then evaluated as in equations (4), (6), (8) and (10). Thus, for the Three-Color Urn, the appropriate state space on which to apply the models is not the *mixed* space {red,black,yellow}, but rather the underlying *purely subjective* space $\{s_0,\ldots,s_{60}\}$ = {0 black balls,…,60 black balls}, whose uncertainty is, after all, the underlying source of the urn's ambiguity.[16] Expressed in this manner, Ellsberg's Three-Color bets take the following form, where in each case i runs from 0 to 60:

$$(11)\qquad \begin{array}{ll} a_1 = \left[\ldots;(\$0,\tfrac{60}{90};\$100,\tfrac{30}{90}) \text{ if } s_i;\ldots\right] & a_2 = \left[\ldots;(\$0,\tfrac{90-i}{90};\$100,\tfrac{i}{90}) \text{ if } s_i;\ldots\right] \\ a_3 = \left[\ldots;(\$0,\tfrac{i}{90};\$100,\tfrac{90-i}{90}) \text{ if } s_i;\ldots\right] & a_4 = \left[\ldots;(\$0,\tfrac{30}{90};\$100,\tfrac{60}{90}) \text{ if } s_i;\ldots\right] \end{array}$$

Each of these models has been shown to be consistent with standard Ellsberg-type preferences in the above and similar examples, each has been formally axiomatized, and each has seen applications in economics.[17]

III. Allais-Type Problems under Purely Subjective Uncertainty

A. Purely Objective Allais-Type Problems

Our first observation is straightforward, has been made before, and is included here only for completeness. It is that, since forms (4), (6), (8) and (10) of the above models imply expected utility preferences over purely objective lotteries, they are directly contradicted by purely objective phenomena such as the Allais Paradox, Common Consequence Effect and Common Ratio Effect.[18]

[16] See the Appendix for a further discussion of this issue.

[17] E.g. Chamberlain (2001), Epstein (2001), Hansen and Sargent (2001), Mukerji and Tallon (2004), Klibanoff, Marinacci and Mukerji (2005), Epstein and Schneider (2010), the papers in Gilboa (2004) and the 2011 Symposium Issue of *Economic Theory* in honor of the 50th anniversary of the Ellsberg Paradox (Ellsberg, et al. (2011)).

[18] See the MacCrimmon and Larsson (1979), Schoemaker (1982) or Machina (1987) for reviews of these effects.

Why did the developers of these models embed expected utility into their forms? In the two-outcome world of the classic Ellsberg Paradoxes, the issue is of little consequence – all objective lotteries over a pair of monetary prizes {\$100,\$0} take the form (\$100,p; \$0,1–p), and given monotonicity with respect to p it is impossible to violate the expected utility hypothesis. It is only when three or more payoffs are allowed that objective expected utility is subject to Allais.

Of course, one could argue that such models were not developed to address Allais-type phenomena under purely objective uncertainty, but rather Ellsberg-type phenomena which inherently involve purely subjective and/or mixed objective/subjective uncertainty.[19] However, it turns out that in a world of three outcomes, these models will be subject to Allais-type difficulties even under purely subjective uncertainty.

B. Purely Subjective Allais-Type Problems

The reason why Allais-type problems can extend to *purely subjective* uncertainty is that some purely subjective events can be said to be "more objective" than others. Take a continuum state space $\mathcal{S}$ = [0,1], partition it into m equal intervals $\{[0,1/m),\ldots,[i/m,(i+1)/m),\ldots,[(m-1)/m,1)\}$,[20] and for each $\alpha \in [0,1]$ define $[0,\alpha)\underset{m}{\times}\mathcal{S}$ as the union of the left α portions of these intervals, so that $[0,\alpha)\underset{m}{\times}\mathcal{S} = \cup_{i=0}^{m-1}[i/m,(i+\alpha)/m)$. As shown by Poincaré (1912) and others,[21]

[19] Even so, it would be nice to achieve a model:phenomenon ratio of less than unity. In any event, see Section V.

[20] For simplicity, we ignore the rightmost state s = 1 in this and subsequent almost-objective partitions and bets.

[21] See Machina (2004, p.9).

such events will satisfy $\lim_{m\to\infty} \pi([0,\alpha)\underset{m}{\times}S) = \alpha$ for any measure $\pi(\cdot)$ over [0,1] with a sufficiently regular density.

More generally, for any set $\wp \subseteq [0,1)$ consisting of a finite union of intervals, and any positive integer *m*, define the event $\wp\underset{m}{\times}S$ by

$$(12) \qquad \wp\underset{m}{\times}S \;=\; \bigcup_{i=0}^{m-1}\{\,(i+\omega)/m \mid \omega\in\wp\}$$

that is, as the union of the natural images of $\wp$ into each of S's equal-length intervals. Events with this type of periodic structure are termed *almost-objective events,* and satisfy $\lim_{m\to\infty} \pi(\wp\underset{m}{\times}S) = \lambda(\wp)$ for uniform Lebesgue measure $\lambda(\cdot)$ over [0,1]. In the limit, agents with "event-smooth" preference functions will treat these events in much the same way as objective events. As $m\to\infty$ such agents will, in the limit, exhibit common outcome-invariant revealed likelihoods of almost-objective events, independence of almost-objective and fixed subjective events, probabilistically sophisticated preferences over almost-objective bets, and for subjective expected utility maximizers, linearity in almost-objective likelihoods and mixtures.[22]

We accordingly posit that as *m* grows large, individuals will converge to indifference between an almost-objective bet $[x_1 \text{ on } \wp_1\underset{m}{\times}S;\ldots;x_n \text{ on } \wp_n\underset{m}{\times}S]$ and its corresponding purely objective lottery $(x_1,\lambda(\wp_1);\ldots;x_n,\lambda(\wp_n))$. Indeed, most so-called "objective" randomizing devices actually generate almost-objective uncertainty: the events {heads,tails} for a 50:50 coin are each periodic events in the (subjective) force of the flip, and each slot on a roulette wheel is periodic in

[22] Machina (2004,Thms. 1,2&5,3,6). The definition of "event-smooth" is that of Machina (2004, pp. 34-36).

the subjective force of the spin, yet they are viewed by decision makers as purely objective.

Given an individual with the standard preference rankings $\mathbf{P}_1 \prec \mathbf{P}_2$ and $\mathbf{P}_3 \succ \mathbf{P}_4$ over the Common Ratio Effect lotteries[23]

$$
(13) \qquad \begin{array}{rcccrcl}
\mathbf{P}_1 & = & (\$6,000,.45\,;\$0,.55) & \prec & \mathbf{P}_2 & = & (\$3,000,.90\,;\$0,.10) \\
\mathbf{P}_3 & = & (\$6,000,.001;\$0,.999) & \succ & \mathbf{P}_4 & = & (\$3,000,.002\,;\$0,.998)
\end{array}
$$

pick $\varepsilon > 0$ small enough so that both $\mathbf{P}_1 \prec \mathbf{P}_2^{\varepsilon} = (\$3{,}000{-}\varepsilon,.90;\$0,.10)$ and $\mathbf{P}_3^{\varepsilon} = (\$6{,}000{-}\varepsilon,.001;\ \$0,.999) \succ \mathbf{P}_4$. Thus, in the Thermometer Problem, they would presumably exhibit the rankings

$$
(14) \qquad \begin{array}{l}
f_{1,m}(\cdot) \;=\; \big[\$6\text{K on } [0,.45) \underset{m}{\times} S\,;\$0 \text{ on } [.45,1) \underset{m}{\times} S\big] \\
\qquad\qquad \prec \; \big[\$3\text{K}{-}\,\varepsilon \text{ on } [0,.90) \underset{m}{\times} S\,;\$0 \text{ on } [.90,1) \underset{m}{\times} S\big] \;=\; f^{\varepsilon}_{2,m}(\cdot) \\
f^{\varepsilon}_{3,}(\cdot) \;=\; \big[\$6\text{K}{-}\,\varepsilon \text{ on } [0,.001) \underset{m}{\times} S\,;\$0 \text{ on } [.001,1) \underset{m}{\times} S\big] \\
\qquad\qquad \succ \; \big[\$3\text{K on } [0,.002) \underset{m}{\times} S\,;\$0 \text{ on } [.002,1) \underset{m}{\times} S\big] \;=\; f_{4,m}(\cdot)
\end{array}
$$

for all m greater than some m_0. Since all such m are finite, all such acts are purely subjective.

To see that such preferences over these purely subjective acts are incompatible with the Multiple Priors form (5), consider a family $\mathcal{P}_0$ of priors $\pi(\cdot)$ over a state space $S = [0,1]$, with uniformly bounded and uniformly continuous densities. By Machina (2004,Thm.0), for each finite interval union $\wp \subseteq [0,1)$, the convergence of $\pi(\wp \underset{m}{\times} S)$ to $\lambda(\wp)$ will be uniform over $\mathcal{P}_0$. We thus have

[23] Kahneman and Tversky (1979, p.267) reported that 86% of their experimental subjects preferred $\mathbf{P}_2$ over $\mathbf{P}_1$, and 73% preferred $\mathbf{P}_3$ over $\mathbf{P}_4$.

$$
(15)\quad
\begin{aligned}
&\lim_{m\to\infty} W_{MP}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} S;\ldots;x_n \text{ on } \wp_n \underset{m}{\times} S\big) \equiv \lim_{m\to\infty}\min_{\pi(\cdot)\in\mathcal{P}_0}\sum\nolimits_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S) = \\
&\min_{\pi(\cdot)\in\mathcal{P}_0}\lim_{m\to\infty}\sum\nolimits_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S) \equiv \min_{\pi(\cdot)\in\mathcal{P}_0}\sum\nolimits_{i=1}^n U(x_i)\cdot\lambda(\wp_i) = \sum\nolimits_{i=1}^n U(x_i)\cdot\lambda(\wp_i)
\end{aligned}
$$

That is to say, as $m \to \infty$, Multiple Priors preferences over almost-objective bets converge to expected utility. Setting $U(\$0) = 0$, (14) and (15) then yield the incompatible inequalities

$$
(16)\quad
\begin{aligned}
.45\cdot U(\$6\text{K}) \quad &= \quad \lim_{m\to\infty} W_{MP}(f_{1,m}(\cdot)) \;\le\; \lim_{m\to\infty} W_{MP}(f^{\varepsilon}_{2,m}(\cdot)) \\
&\qquad\qquad\qquad = \;\; .90\cdot U(\$3\text{K}-\varepsilon) \;<\; .90\cdot U(\$3\text{K}) \\
.001\cdot U(\$6\text{K}) \quad &> \quad .001\cdot U(\$6\text{K}-\varepsilon) \;=\; \lim_{m\to\infty} W_{MP}(f^{\varepsilon}_{3,m}(\cdot)) \\
&\qquad\qquad\qquad \ge \;\; \lim_{m\to\infty} W_{MP}(f_{4,m}(\cdot)) \;=\; .002\cdot U(\$3\text{K})
\end{aligned}
$$

To see that such preferences are also incompatible with the Smooth Ambiguity form (7), assume enough regularity so that the limit can be moved inside both the integral sign and $\phi(\cdot)$, so that

$$
(17)\quad
\begin{aligned}
&\lim_{m\to\infty} W_{SM}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} S;\ldots;x_n \text{ on } \wp_n \underset{m}{\times} S\big) \quad = \\
&\lim_{m\to\infty}\int_{\pi(\cdot)\in\mathcal{P}} \phi\big(\sum\nolimits_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S)\big)\cdot d\mu(\pi(\cdot)) \quad = \\
&\int_{\pi(\cdot)\in\mathcal{P}} \phi\big(\lim_{n\to\infty}\sum\nolimits_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S)\big)\cdot d\mu(\pi(\cdot)) \quad = \\
&\int_{\pi(\cdot)\in\mathcal{P}} \phi\big(\sum\nolimits_{i=1}^n U(x_i)\cdot\lambda(\wp_i)\big)\cdot d\mu(\pi(\cdot)) \quad \equiv \quad \phi\big(\sum\nolimits_{i=1}^n U(x_i)\cdot\lambda(\wp_i)\big)
\end{aligned}
$$

That is to say, as $m \to \infty$, this model's preferences over almost-objective bets also converge to expected utility.[24] Defining $f_{1,m}(\cdot)$, $f^{\varepsilon}_{2,m}(\cdot)$, $f^{\varepsilon}_{3,m}(\cdot)$ and $f_{4,m}(\cdot)$ as in (14) and setting $U(\$0) = 0$ again yields incompatible inequalities:

[24] This has also been observed by Klibanoff, Marinacci and Mukerji (2005,p.1855,n.3).

$$
(18)\quad
\begin{aligned}
&\phi\big(.45\cdot U(\$6\text{K})\big) \;=\; \lim_{m\to\infty} W_{SM}(f_{1,m}(\cdot)) \;\le\; \lim_{m\to\infty} W_{SM}(f^{\varepsilon}_{2,m}(\cdot)) \;=\\
&\qquad\qquad\qquad\qquad \phi\big(.90\cdot U(\$3\text{K}-\varepsilon)\big) \;<\; \phi\big(.90\cdot U(\$3\text{K})\big)\\
&\phi\big(.001\cdot U(\$6\text{K})\big) \;>\; \phi\big(.001\cdot U(\$6\text{K}-\varepsilon)\big) \;=\; \lim_{m\to\infty} W_{SM}(f^{\varepsilon}_{3,m}(\cdot)) \;\ge\\
&\qquad\qquad\qquad\qquad \lim_{m\to\infty} W_{SM}(f_{4,m}(\cdot)) \;=\; \phi\big(.002\cdot U(\$3\text{K})\big)
\end{aligned}
$$

To see that such preferences are also incompatible with the Variational Preferences form (9), assume enough regularity so that the limit can be moved inside the minimum function, to get

$$
(19)\quad
\begin{aligned}
&\lim_{m\to\infty} W_{VP}\big(x_1 \text{ on } \wp_1 \underset{m}{\times} S;\ldots;x_n \text{ on } \wp_n \underset{m}{\times} S\big) =\\
&\lim_{m\to\infty}\ \min_{\pi(\cdot)\in\mathcal{P}}\Big(\textstyle\sum_{i=1}^{n} U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S)+c\big(\pi(\cdot)\big)\Big)\\
&= \min_{\pi(\cdot)\in\mathcal{P}}\Big(\textstyle\sum_{i=1}^{n} U(x_i)\cdot \lim_{m\to\infty}\pi(\wp_i \underset{m}{\times} S)+c\big(\pi(\cdot)\big)\Big) =\\
&\min_{\pi(\cdot)\in\mathcal{P}}\Big(\textstyle\sum_{i=1}^{n} U(x_i)\cdot\lambda(\wp_i)+c\big(\pi(\cdot)\big)\Big) \;=\; \textstyle\sum_{i=1}^{n} U(x_i)\cdot\lambda(\wp_i) \;+\; \min_{\pi(\cdot)\in\mathcal{P}} c\big(\pi(\cdot)\big)
\end{aligned}
$$

Since the "min" term in the final expression is a constant independent of both the x_i's and $\wp_i$'s of an almost-objective prospect, such preferences again converge to expected utility over such prospects, and are thus incompatible with the Common Ratio Effect preferences (14).

IV. Attitudes Toward Different Sources (and Amounts) of Ambiguity

Although a three-outcome world leaves the Multiple Priors, Smooth Ambiguity and Variational Preferences Models subject to purely subjective Allais-type effects, nonadditivity of the measure $C(\cdot)$ in equations (3) and (4) implies that Rank-Dependent preferences are not necessarily subject to such effects. However, allowing for three or more outcomes does present a different type of difficulty for this model.

In bets that involve only two outcomes, such as the classic Ellsberg examples, the outcome values are necessarily "adjacent" to each other. However, adding a third possible outcome to an Ellsberg-type bet allows for possible interactions between outcome values that are not adjacent, such as the outcomes +\$8,000 and –\$8,000 in the Slightly Bent Coin Problem.

The Slightly Bent Coin Problem differs from a typical Allais- or Ellsberg-type problem (as well as from the examples in Machina (2009)) in that it involves two rather than four bets, neither of which is purely objective or even involves any purely objective events. In the author's view, the ambiguity-averse choice would be for Bet I, which spreads the uncertainty of receiving +\$8,000 versus –\$8,000 across the less ambiguous coin rather than the more ambiguous ball. Others have told me they view Bet II as less ambiguous. A strict preference in either direction would violate the Subjective Expected Utility Model: since informational symmetry would imply $\mu(BH) = \mu(BT) = \mu(WH) = \mu(WT) = ¼$, both bets would have a common subjective expected utility of $¼ \cdot U(+\$8{,}000) + ½ \cdot U(\$0) + ¼ \cdot U(-\$8{,}000)$.

Although the ambiguity averse ranking Bet I $\succ$ Bet II is consistent with the Multiple Priors, Smooth Ambiguity and Variational Preferences Models,[25] neither it nor the reversed ranking Bet I $\prec$ Bet II is consistent with the Rank-Dependent

[25] For the Multiple Priors Model, letting each prior $\pi(\cdot)$ in $\mathcal{P}_0$ exhibit independence of the coin and urn, with $\{(\pi(\text{black}), \pi(\text{heads}))\} = [\varepsilon, 1-\varepsilon] \times [½-\varepsilon, ½+\varepsilon]$ for sufficiently small ε, and letting $U(\cdot)$ be sufficiently concave would yield W_{MP}(Bet I) > W_{MP}(Bet II). For the Smooth Ambiguity Model, adding the assumption that $\mu(\cdot)$ is the uniform measure over the set $[\varepsilon, 1-\varepsilon] \times [½-\varepsilon, ½+\varepsilon]$ would also yield W_{SM}(Bet I) > W_{SM}(Bet II). For the Variational Preferences Model, use the special case when it reduces to Multiple Priors with the above properties.

Model, which implies that the two bets *must be indifferent* to each other. Although their good news events BH and WT for the payoff +\$8,000 are not the same, they are informationally symmetric, which would presumably imply $C(BH) = C(WT)$. The two bets *do* have the same good news event for the payoff \$0, namely $BH{\cup}WH{\cup}WT$. Together, this implies that the values

$$
(20)\qquad
\begin{aligned}
W_{RD}(\text{B}{\small\text{ET}}\ 1) &= U(+8K)\cdot C(BH) + U(0)\cdot[C(BH{\cup}WH{\cup}WT) - C(BH)] \\
&\qquad + U(-8K)\cdot[1 - C(BH{\cup}WH{\cup}WT)] \\
W_{RD}(\text{B}{\small\text{ET}}\ 2) &= U(+8K)\cdot C(WT) + U(0)\cdot[C(BH{\cup}WH{\cup}WT) - C(WT)] \\
&\qquad + U(-8K)\cdot[1 - C(BH{\cup}WH{\cup}WT)]
\end{aligned}
$$

must be equal, so that the Rank-Dependent Model implies indifference between the bets. In other words, the Rank-Dependent Model cannot represent a preference (in either direction) for one of these sources of ambiguity over the other. This is presumably relevant if real-world decisions involve different sources, with different amounts, of ambiguity.[26]

To see that this difficulty arises from the triple of outcome values $\{+\$8{,}000, \$0, –\$8{,}000\}$, replace the \$0's by +\$8,000 and –\$8,000 to obtain the adjacent-outcome bets Bet I* and Bet II* below. An ambiguity averter would

[26] Why a *bent* coin? Why not use an objective 50:50 coin to make the same point? As noted by Lo (2008), such a version appears in a manuscript version of Machina (2009). (Lo applies this version to the model of Klibanoff (2001); Baillon, L'Haridon and Placido (2011) and Blavatskyy (2012) apply it to other models.) We use a bent (and therefore purely subjective) coin here in order to show that the difficulties with the Rank-Dependent Model appear even in a setting of purely subjective uncertainty – that is, directly from equation (3), and not solely in the mixed uncertainty domain of equation (4).

presumably still prefer to stake the ±\$8,000 uncertainty on the less ambiguous coin (Bet I*) than on the more ambiguous ball color (Bet II*), and the Rank-Dependent Model can represent this preference by simply positing $C(BH \cup WH) > C(WH \cup WT)$. The model captures ambiguity aversion in Ellsberg's examples by inequalities such as $C(A \cup B) > C(A) + C(B)$ whenever the union of two ambiguous events A and B is an objective event, such as black and yellow in the Three-Color Ellsberg urn. But these examples only involve unions of events with adjacent outcome values (\$0 and \$100 for Ellsberg, and ±\$8,000 for Bets I* and II*). The Slightly Bent Coin Problem illustrates that the Rank-Dependent Model may not be able to capture attitudes toward different sources or amounts of ambiguity when it is across nonadjacent outcome values.

	Bet I*				Bet II*	
	black	white			black	white
heads	+\$8,000	+\$8,000	vs.	heads	–\$8,000	+\$8,000
tails	–\$8,000	–\$8,000		tails	–\$8,000	+\$8,000

In fact, the difficulties raised by the two-source Slightly Bent Coin Problem extend to the case of just a single source of subjective uncertainty. Say you've learned that a meteor of unknown size or speed has been spotted, is predicted to strike the earth sometime the day after tomorrow, and you are betting on a single subjective variable, namely the longitude ℓ of its strike. Since you know nothing more, from your point of view the circular state space $\mathcal{S} = [0^{\circ},360^{\circ})$ is informationally symmetric. The following bets are based on whether ℓ lands in the interval $[0^{\circ},180^{\circ})$ or $[180^{\circ},360^{\circ})$, and whether its seventh decimal is even or odd. This is seen to be a single-source analog of the Slightly Bent Coin Problem, where its highly subjective ball is replaced by the highly subjective partition $\{[0^{\circ},180^{\circ}),[180^{\circ},360^{\circ})\}$, and its slightly bent coin is replaced by the almost-objective partition $\{7^{\text{th}} \text{ decimal even}, 7^{\text{th}} \text{ decimal odd}\}$.[27] Presumably, an ambiguity averse decision maker would prefer to spread the $\pm$\$8,000 uncertainty across the almost-objective 7th decimal than across the much more subjective hemispheres, and prefer Bet I over Bet II. However, the good news events for the payoff \$8,000 are informationally symmetric for the two bets, and their good news events for \$0 are again identical, so the Rank-Dependent Model cannot reveal a preference for one of these forms of ambiguity over the other.

METEORITE PROBLEM (ℓ = LONGTITUDE OF FIRST STRIKE)

	BET I				BET II	
	$\ell \in [0^{\text{o}},180^{\text{o}})$	$\ell \in [180^{\text{o}},360^{\text{o}})$			$\ell \in [0^{\text{o}},180^{\text{o}})$	$\ell \in [180^{\text{o}},360^{\text{o}})$
7th decimal even	+\$8,000	\$0	vs.	7th decimal even	\$0	\$0
7th decimal odd	−\$8,000	\$0		7th decimal odd	−\$8,000	+\$8,000

V. Attitudes Toward Ambiguity at Different Outcome Levels

A third aspect of ambiguity aversion which can arise in a world of three or more outcomes is that, just as with risk and risk aversion, ambiguity can occur at different final wealth levels or different gain/loss levels in a prospect, and individuals may exhibit different amounts of ambiguity aversion at these different outcome levels. This is not so apparent in the three-outcome bets of the Slightly-Bent Coin Problem, where in each bet its upper and lower outcomes +\$8,000 and

[27] Since it is based on only a single source of subjective uncertainty, this example also differs from the examples in Machina (2009), which are based on one objective and two separate sources of subjective uncertainty.

–\$8,000 enter with equal ambiguity. But in general, a prospect with three or more outcomes can exhibit more ambiguity at or about some outcome levels than others, and this leads to another source of difficulties for the four models we're considering. This can occur whether or not the functions $U(\cdot)$ and $\phi(\cdot)$ in these form exhibit the standard Arrow-Pratt conditions.[28] As in Section IV, each example in this section only involves a single pair of bets.

This can be seen from the Upper/Lower Tail Problem. Each urn's bet involves the same triple of outcomes \$0, \$$c$ and \$100, where \$$c$ is the decision maker's certainty equivalent of the objective lottery (\$0,½;\$100,½).[29] But as noted above, the ambiguity in Urn I is across the lower outcomes \$0 and \$$c$, whereas in Urn II it is across the upper outcomes \$$c$ and \$100. If ambiguity aversion somehow involves "pessimism," mightn't an ambiguity averter have a strict preference for Urn II over Urn I?

Maybe yes, maybe no. But whether or not an ambiguity averter *should* prefer one bet over the other, none of the models we are considering *allow* this to happen in either direction. To see this, express these mixed bets in a manner which separates their objective from their subjective uncertainty – that is, as Anscombe-Aumann acts – so they can be evaluated by the models' formulas (4), (6), (8) and (10). The following table presents these mappings from states to objective lotteries, where the underlying subjective uncertainty of each urn is given by its

[28] That is to say, whether or not the expressions $-U''(x)/U'(x)$ or $-\phi''(x)/\phi'(x)$ are constant, increasing or decreasing in x.

[29] Lest the value \$$c$ in this example be interpreted as coming from backwards induction of some two-stage prospects, we assume that this monetary amount has been independently and previously obtained from preferences over single-stage, purely objective lotteries.

state space {*BB*,*BW*,*WB*,*WW*}, namely the Cartesian product of the informationally symmetric spaces {ball #1 black, ball #1 white} and {ball #2 black, ball #2 white}. For example, when both the unknown-color balls in Urn I are black, it yields a 2/3 chance of paying \$0 and a 1/3 chance of paying \$100; if both are white, it yields a 2/3 chance of \$*c* and a 1/3 chance of \$100, etc.

	BB	*BW*	*WB*	*WW*
URN I	(\$0,⅔;\$100,⅓)	(\$0,⅓;\$*c*,⅓;\$100,⅓)	(\$0,⅓;\$*c*,⅓;\$100,⅓)	(\$*c*,⅔;\$100,⅓)
URN II	(\$0,⅓;\$*c*,⅔)	(\$0,⅓;\$*c*,⅓;\$100,⅓)	(\$0,⅓;\$*c*,⅓;\$100,⅓)	(\$0,⅓;\$100,⅔)
expected utility	⅓	½	½	⅔

If we normalize so that $U(\$0) = 0$ and $U(\$100) = 1$, \$*c* will have a von Neumann-Morgenstern utility value of ½, and equations (4), (6), (8) and (10) imply

Rank-Dependent Model (*equation* (4)):

$$(21)\qquad \begin{aligned} W_{RD}(\text{URN I}) \;&=\; \tfrac{2}{3}\cdot C(WW) \;+\; \tfrac{1}{2}\cdot[C(WW\cup BW\cup WB)-C(WW)] \;+\\ &\quad \tfrac{1}{3}\cdot[1-C(WW\cup BW\cup WB)] \;=\; W_{RD}(\text{URN II}) \end{aligned}$$

Multiple Priors Model (*equation* (6)):

$$(22)\qquad \begin{aligned} W_{MP}(\text{URN I}) \;&=\; \min_{(p_{BB},p_{BW},p_{WB},p_{WW})\in\mathcal{P}_0}\left[\tfrac{1}{3}\cdot p_{BB}+\tfrac{1}{2}\cdot p_{BW}+\tfrac{1}{2}\cdot p_{WB}+\tfrac{2}{3}\cdot p_{WW}\right] \\ &=\; W_{MP}(\text{URN II}) \end{aligned}$$

Smooth Ambiguity Model (*equation* (8)):

$$(23)\qquad \begin{aligned} &W_{SM}(\text{URN I}) \;=\\ &\int\phi\left(\tfrac{1}{3}\cdot p_{BB}+\tfrac{1}{2}\cdot p_{BW}+\tfrac{1}{2}\cdot p_{WB}+\tfrac{2}{3}\cdot p_{WW}\right)\cdot d\,\mu(p_{BB},p_{BW},p_{WB},p_{WW}) \\ &\quad =\; W_{SM}(\text{URN II}) \end{aligned}$$

Variational Preferences Model (*equation* (10)):

$$
\begin{aligned}
&W_{VP}(\text{URN I}) \quad = \\
(24) \qquad &\min_{(p_{BB},p_{BW},p_{WB},p_{WW})\in\mathcal{P}} \left[\tfrac{1}{3}\cdot p_{BB} + \tfrac{1}{2}\cdot p_{BW} + \tfrac{1}{2}\cdot p_{WB} + \tfrac{2}{3}\cdot p_{WW} + c(p_{BB},p_{BW},p_{WB},p_{WW}) \right] \\
&\qquad\qquad = \quad W_{VP}(\text{URN II})
\end{aligned}
$$

In other words, none of the four models allow the decision maker to exhibit an aversion to ambiguity in the lower tail relative to ambiguity in the upper tail, or vice versa, in the Upper/Lower Tail Problem.

What is it in the structure of these bets that necessitates this indifference? As seen from the first two rows of the above table, the two bets have different mappings from states to objective lotteries, which is the source of their different ambiguity structures. But as seen in the third row, they have the *same statewise distribution of expected utility values.* Any decision model which evaluates a bet solely on its mapping from states to these values – as do the Rank-Dependent, Multiple Priors, Smooth Ambiguity and Variational Preferences Models[30] – must necessarily be indifferent between the two bets, in spite of the fact that their ambiguity occurs at different outcome levels.

Besides attitudes toward ambiguity in the lower versus upper outcomes of a prospect, there are other aspects of ambiguity aversion which can arise in a world of three or more outcomes but which cannot be fully captured by the four models. A relative preference for (or aversion to) ambiguity in central outcomes versus ambiguity in tail outcomes could be exhibited by an individual's ranking of the prospects

[30] This follows since, for each event E_i, its conditional objective lottery $(\ldots;x_{ij},p_{ij};\ldots)$ only enters via the bracket term $[\sum_j U(x_{ij})\cdot p_{ij}]$ in each of equations (4), (6), (8) and (10).

SPREADS IN THE AMBIGUITY OVER A FIXED SET OF OUTCOMES

URN I							URN II					
one ball	two balls		two balls		one ball		two balls		one ball	one ball	two balls	
purple	black	white	red	green	yellow	vs.	black	white	purple	yellow	red	green
1	2	3	4	5	6		1	2	3	4	5	6

where in this and the following examples, outcomes are expressed in utils rather than dollars.[31] Again, the two urns' bets involve the same set of outcome values, but Urn I's ambiguity is in its central outcome values (2 vs. 3 and 4 vs. 5), whereas Urn II's ambiguity is in its tails (1 vs. 2 and 5 vs. 6). It is straightforward to show that the two bets again exhibit the same mapping from states to expected utility, so none of the four models can exhibit a strict preference one way or the other.

Instead of spreads in the ambiguity over a fixed set of outcomes, we can consider spreads in the outcomes for a fixed ambiguity structure. This can be exhibited by the prospects:

SPREADS IN THE OUTCOMES FOR A FIXED AMBIGUITY STRUCTURE

BET I					BET II			
one ball		one ball			one ball		one ball	
black	white	red	green	vs.	black	white	red	green
5	6	7	8		1	2	11	12

These two prospects possess the same ambiguity structure, and each involves ambiguity in both its upper and lower tail. However Bet I has a smaller outcome

[31] As with the value $\$c$ in Note 29, we assume that these utility values have been previously obtained from single-stage objective lotteries.

spread than Bet II. Since the outcomes are in utils, any subjective expected utility maximizer whose prior reflects the informational symmetry of the urns would be indifferent. However, if it is the distribution and not just the mean of utility that matters, an ambiguity averter may well express a preference for low-spread versus high-spread prospects. But as before, the two bets imply the same mapping from states to expected utility, so none of the four models can allow for a strict preference.

One might argue that a preference for Bet I over Bet II in this example would not be a feature of ambiguity preferences at all, but rather, implied by an individual's risk aversion over monetary outcomes. But since the outcomes are expressed in utils, any subjective expected utility maximizer would be indifferent between them, regardless of the concavity/convexity of their utility of money function, and hence regardless of their attitudes toward risk. Any departure from indifference must be due to attitudes toward the differing ambiguity (in this case, its differing location) in the prospects – attitudes which cannot be captured by either subjective expected utility or the major models.

Perhaps the most important feature of ambiguity preferences that can arise in a world of three or more outcomes is that attitudes toward ambiguity at high versus low final wealth levels, or in gains versus losses, may differ. Although this is to some extent displayed by the Upper/Lower Tail Problem, it is worth more explicit treatment. Consider Bets I and II in the following table. Bet I differs from a 50:50 objective bet with payoffs {4,8} by introducing ambiguity about its low outcome 4, whereas Bet II differs by introducing ambiguity about its high outcome 8.

AMBIGUITY AT HIGH VERSUS LOW OUTCOME LEVELS

BET I					BET II			
one ball		one ball			one ball		one ball	
black	white	red	green	vs.	black	white	red	green
3	5	8	8		7	9	4	4

Again, informational symmetry implies that a subjective expected utility maximizer would have equal subjective probabilities over the states $\{BR,BG,WR,WG\}$, and accordingly be indifferent. Properly reordered,[32] their outcome sets {3,5,8,8} and {7,9,4,4} are reflections about the values {6,6,6,6}, so neither prospect can be said to have a greater outcome spread than the other. But if ambiguity about a final wealth level of 4 matters differently than ambiguity about a final wealth level of 8, an ambiguity averter may have a strict preference for one bet over the other. But again, the two urns have identical mappings from states to expected utility levels, so the four models must again rank them as indifferent – none of them can allow for phenomena such as "decreasing absolute ambiguity aversion."

Starting from a purely objective 50:50 bet over the payoffs {–2,+2} yields an equivalent version of the example, this time involving attitudes toward ambiguity in losses versus gains.

AMBIGUITY IN LOSSES VERSUS GAINS

BET I					BET II			
one ball		one ball			one ball		one ball	
black	white	red	green	vs.	black	white	red	green
–3	–1	+2	+2		+1	+3	–2	–2

[32] Choose the informationally symmetric reordering {9,7,4,4}.

In Section III we noted that the conditional expected utility terms $\sum_j U(x_{ij})\cdot p_{ij}$ in forms (4), (6), (8) and (10) left these models susceptible to Allais-type problems under purely objective uncertainty, and we have just seen how prospects with identical mappings from states to expected utility levels can nevertheless have very different ambiguity properties. It might seem that both problems could be averted by replacing the conditional expected utility terms in these forms by a Quiggin (1982) type *anticipated utility* term $\sum_j U(x_{ij})\cdot(C(\cup_{k=1}^{j} p_{ik}) - C(\cup_{k=1}^{j-1} p_{ik}))$ with appropriate choice of capacity $C(\cdot)$.[33] Although doing so may or may not avert objective Allais-type difficulties,[34] it cannot avert the types of difficulties presented in this section.

To see this, define $\hat{p}$ so that $C(\hat{p})$ = ½. Expressing outcomes in utils, the objective bet (–2,1–$\hat{p}$;+2,$\hat{p}$) plays the same role for anticipated utility preferences as does the objective bet (–2,½;+2,½) for expected utility preferences: it is a two-outcome bet with symmetric and equally weighted gains and losses, which is indifferent to a sure payoff of 0. Consider the following modification of the above Loss versus Gains example, with its ½:½ objective probabilities of drawing a black/white versus red/green ball replaced by an objective (1–$\hat{p}$):$\hat{p}$ coin flip to determine from which of two urns a ball is to be drawn. Bet I differs from the objective bet (–2,1–$\hat{p}$;+2,$\hat{p}$) by introducing ambiguity about its low outcome level –2, whereas Bet II differs by introducing ambiguity about its high outcome

[33] See, for example, Klibanoff, Marinacci and Mukerji (2005,p.1859). Along the lines of equations (3) and (4), the variables are labeled such that the terms $\sum_j U(x_{ij})\cdot(C(\cup_{k=1}^{j} p_{ik}) - C(\cup_{k=1}^{j-1} p_{ik}))$ are decreasing in i, and for each i, the values x_{ij} are decreasing in j.

[34] E.g., Wu (1994), Birnbaum (2006,2008).

level +2. Below are the bets' statewise conditional objective lotteries and their corresponding anticipated utility values.

AMBIGUITY IN LOSSES VERSUS GAINS (MODIFIED VERSION)

BET I					BET II			
prob $(1-\hat{p})$		prob $\hat{p}$			prob $\hat{p}$		prob $(1-\hat{p})$	
URN I (1 ball)		URN II (1 ball)			URN I (1 ball)		URN II (1 ball)	
black	white	red	green	vs.	black	white	red	green
–3	–1	+2	+2		+1	+3	–2	–2

black:red	black:green	white:red	white:green	black:red	black:green	white:red	white:green
$(2,\hat{p};-3,1-\hat{p})$	$(2,\hat{p};-3,1-\hat{p})$	$(2,\hat{p};-1,1-\hat{p})$	$(2,\hat{p};-1,1-\hat{p})$	$(1,\hat{p};-2,1-\hat{p})$	$(1,\hat{p};-2,1-\hat{p})$	$(3,\hat{p};-2,1-\hat{p})$	$(3,\hat{p};-2,1-\hat{p})$
–½	–½	½	½	–½	–½	½	½

In this case, the bets have the same mappings from states to conditional anticipated utility levels, so forms which evaluate prospects solely on the basis of these mappings must again rank the bets as indifferent, and cannot admit phenomena such as decreasing/increasing absolute ambiguity aversion.

VI. Conclusion

The examples of this paper show that allowing three or more outcome values raises aspects of ambiguity and ambiguity aversion which do not appear in the classic two-outcome Ellsberg examples, and are not always captured by models developed in response to these original examples. Although these newer examples take a few different forms, they all arise from separability properties of the classic Subjective Expected Model which are retained by these models.

Oddly enough, the Allais-type difficulties posed by Section III's Thermometer Problem do *not* arise from the manner in which the Multiple Priors, Smooth Preferences or Variational Preferences Models treat purely objective or mixed objective:subjective uncertainty. Rather, these difficulties were seen to follow directly from the models' purely subjective forms (5), (7) and (9), and in particular, how they evaluate the purely subjective family of almost-objective acts. In the limit, a value $\pi(\wp \underset{m}{\times} S)$ will converge to $\lambda(\wp)$ regardless of the particular measure $\pi(\cdot)$, and when $\pi(E_i)$ is replaced by $\lambda(E_i)$ in any of the forms (5), (7) or (9), it reverts to the additively separable expected utility form. In other words, models which retain too much separability over purely subjective uncertainty are subject to analogues of *purely objective* Allais-type problems. This is why the Rank-Dependent form (3), which does not share this feature, is not subject to such difficulties.

The Meteorite Problem of Section IV arises from the fact that with three or more outcomes, there exist subjective partitions and purely subjective prospects over a state space which differ in their ambiguity structure, but not in their assignment of (ranked) outcomes to subjective events – or rather, in their assignment of ranked outcomes to informationally equivalent subjective events. Since the Subjective Expected Utility Model depends solely and separably upon this assignment, it cannot capture attitudes toward these different ambiguity structures. The Meteorite Problem shows that even though the Rank-Dependent Model only retains separability across similarly ranked prospects, that is enough to leave it subject to the same problems.

The Upper/Lower Tail Problem and other examples of Section V arise from the fact that all four models evaluate the conditional objective lotteries of a mixed objective/subjective prospect via their expected utilities, in a manner which is conditionally separable[35] across states – this, after all, is what allows us to

[35] By "conditionally separable" I mean independently of the conditional objective lotteries of other states.

represent the prospects of Section V as mappings from states to expected utility values. The examples of this section show that in a world of three or more outcomes, conditional objective lotteries can differ from each other yet still have the same expected utility, so that one can construct mixed objective/subjective lotteries which differ (sometimes widely) in their ambiguity structure, but whose statewise conditional lotteries have the same expected utilities. In such cases the models are again incapable of capturing attitudes toward these different ambiguity structures.

Separability is practically inherent in any well-specified functional form, or in axioms structures which generate such forms, and the examples of this paper show how the separability properties of the major models of ambiguity aversion can render them subject to the same types of difficulties faced by classic objective expected utility in Allais-type problems, or by classic subjective expected utility in Ellsberg-type problems. This suggests that researchers step back a bit and try to model attitudes toward ambiguity via *curvature conditions* on general preference functionals of the form $V(\ldots;x_i \text{ on } E_i;\ldots)$ or $V(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots)$,[36] rather than the kinds of *separability conditions* inherent in forms (3) through (10).

[36] By "curvature conditions" I mean the direction(s) in which ambiguity averse preferences "bend away" from classical Subjective Expected Utility preferences, much like risk aversion over objective lotteries has been modeled by von Neumann-Morgenstern utility functions which bend away from linearity in the direction of concavity. After all, some of the most fundamental results in economics – the Slutsky equation, existence and efficiency of competitive equilibrium, factor price equalization – are not based on specific functional forms, merely curvature conditions such as quasiconcavity or convexity.

Appendix: What's the Proper State Space for an Ellsberg Urn?

As noted above, there is more than one way to specify the underlying state space of an Ellsberg urn.[37] In our discussion of the classic Three-Color Paradox with its 60 balls of unknown color, we specified a 61-element space, with each state corresponding to the number of black balls. In our discussion of the Upper/Lower Tail Problem, with its two balls of unknown color, we specified a 4-element space of the form {*BB*,*BW*,*WB*,*WW*}. Had we instead defined states by the number of black balls, we would have had a 3-element state space {0 black balls, 1 black ball, 2 black balls}. The relationship between the two specifications is that the latter space is slightly coarser that the former, with its state "1 black ball" being the union of the states *BW* and *WB*.

Does the choice of approach matter? Not always – given informational symmetry, the bets in the Two-Urn and Four-Color Ellsberg Paradoxes are such that the lessons of these examples would be the same under either specification.

However, the two approaches can yield different predictions over other types of bets, even under informational symmetry. Consider, as does Epstein (2010, p.2088), bets on the actual composition of an Ellsberg urn – say Urn I in the Upper/Lower Tail Problem. Informational symmetry of the state space {*BB*,*BW*,*WB*,*WW*} would lead to indifference between staking a prize on the two balls being the same color versus the two balls being of different colors. On the other hand, informational symmetry of the space {0 black balls, 1 black ball, 2 black balls} would lead to a strict preference for the first bet. This distinction is magnified when applied to the 90-ball Three-Color Urn.

[37] See also the discussions in Wakker (2010, Sects. 4.9 and 10.7) and Machina (2011).

Whether individuals treat the states $\{BB, BW, WB, WW\}$ or the states {0 black balls, 1 black ball, 2 black balls} as informationally symmetric is ultimately an empirical question. It turns out that the distinction does not matter for our analysis of the Upper/Lower Tail Problem or any of our other examples. As mentioned, we explicitly adopted the state space $\{BB, BW, WB, WW\}$ in our analysis of this example. However, if we instead adopt the space {0 black balls, 1 black ball, 2 black balls}, the table preceding equations (21) through (24) would take the form

	0 black balls	1 black ball	2 black balls
URN I	$(\$c,⅔;\$100,⅓)$	$(\$0,⅓;\$c,⅓;\$100,⅓)$	$(\$0,⅔;\$100,⅓)$
URN II	$(\$0,⅓;\$100,⅔)$	$(\$0,⅓;\$c,⅓;\$100,⅓)$	$(\$0,⅓;\$c,⅔)$
expected utility	⅔	½	⅓

and the equations themselves would take the forms

Rank-Dependent Model:

(21)*
$$\begin{aligned} W_{RD}(\text{URN I}) \;&=\; \tfrac{2}{3}\cdot C(0\text{ black}) \;+\; \tfrac{1}{2}\cdot[C(0\text{ black}\cup 1\text{ black}) - C(0\text{ black})] \\ &\quad + \tfrac{1}{3}\cdot[1 - C(0\text{ black}\cup 1\text{ black})] \;=\; W_{RD}(\text{URN II}) \end{aligned}$$

Multiple Priors Model:

(22)*
$$\begin{aligned} W_{MP}(\text{URN I}) \;&=\; \min_{(p_{0\text{ black}},\,p_{1\text{ black}},\,p_{2\text{ black}})\in\mathcal{P}_0}\left[\tfrac{2}{3}\cdot p_{0\text{ black}} + \tfrac{1}{2}\cdot p_{1\text{ black}} + \tfrac{1}{3}\cdot p_{2\text{ black}}\right] \\ &=\; W_{MP}(\text{URN II}) \end{aligned}$$

Smooth Ambiguity Model:

(23)*
$$\begin{aligned} W_{SM}(\text{URN I}) \;&=\; \int \phi\left(\tfrac{2}{3}\cdot p_{0\text{ black}} + \tfrac{1}{2}\cdot p_{1\text{ black}} + \tfrac{1}{3}\cdot p_{2\text{ black}}\right)\cdot d\mu(p_{0\text{ black}}, p_{1\text{ black}}, p_{2\text{ black}}) \\ &=\; W_{M}(\text{URN II}) \end{aligned}$$

Variational Preferences Model:

$$W_{VP}(\text{URN I}) \quad = $$

$$(24)^{*} \quad \min_{(p_{0\,\text{black}},\, p_{1\,\text{black}},\, p_{2\,\text{black}}) \in \mathcal{P}} \left[\tfrac{2}{3} \cdot p_{0\,\text{black}} + \tfrac{1}{2} \cdot p_{1\,\text{black}} + \tfrac{1}{3} \cdot p_{2\,\text{black}} + c(p_{0\,\text{black}}, p_{1\,\text{black}}, p_{2\,\text{black}}) \right]$$

$$= \quad W_{VP}(\text{URN II})$$

In other words, the two urns will continue to have the same mapping from states to expected utility levels, and the four models will continue to imply indifference. A similar argument applies to the “Spreads in the Ambiguity over a Fixed Set of Outcomes” example of Section V, which is the only other example in Sections III, IV or V which allows for the two alternative specifications of its state space.

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