Risk Neutral Equilibria of Non-cooperative Games

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Abstract

Game-theoretic concepts such as Nash and Bayesian equilibrium describe and predict strategic behavior in terms of uniquely determined and commonly known probability distributions over the identities and moves of the players. However, there are many potential sources of difficulty in determining the players' reciprocal beliefs: indeterminacy or unobservability of each other's payoff functions, non-uniqueness of equilibria, heterogeneity of prior probabilities, unobservable background risk, and distortions of revealed beliefs due to risk aversion, among others. This paper presents a unified approach for dealing with these issues in which a typical solution of a game is a convex set of probability distributions that, unlike Nash equilibria, may be correlated between players. In the most general case, where players are risk averse, the parameters of equilibria are risk neutral probabilities, interpretable as products of subjective probabilities and relative marginal utilities for money, as in financial markets.

Keywords: coherence, previsions, lower and upper probabilities, Nash equilibrium correlated equilibrium, risk neutral probabilities, risk neutral equilibrium

1 Introduction

Game theory occupies the increasingly large middle ground of rational choice theory: the problem of "2, 3, 4… bodies" in which agents must reason about the strategic behavior of other rational agents as well as reflect on their own preferences and compete in markets. The modeling of interactive decisions of this kind requires some special tools and assumptions. First, the rules of the game are (in the most general case) parameterized in units of utility rather than money or material goods in order to allow for differences in tastes and attitudes toward risk. Second, the utility functions of different players are assumed to be common knowledge, enabling them to model each other's decisions as well as their own, and to all know that they can all do this, and so on. Third, common knowledge of rationality and common knowledge of the rules of the game are assumed to lead to an equilibrium, usually a Nash equilibrium or one of its refinements or extensions, in which the decision of each player is individually rational given the decisions simultaneously made by the other players, and randomization (if any) is performed independently. And fourth, when there is uncertainty about any of the game parameters, the beliefs of the players are assumed to be consistent with a common prior distribution, which generates an infinite hierarchy of mutually consistent reciprocal beliefs. In applications these assumptions are usually applied at maximum strength in order to tightly (often uniquely) constrain the solution, yet all of them are open to question. This paper will pursue some of these questions and show how they lead to solutions that are characterized by exactly the same rationality conditions as individual decisions and competitive markets. Their common priors and equilibria are generally expressed in terms of lower and upper bounds on probabilities that need not satisfy an independence condition and do not always represent the players' true subjective beliefs.

The approach to modeling games that will be used in this paper follows that of Nau and McCardle (1990) and Nau (1992), which is a multi-player extension of de Finetti's operational approach to defining subjective probabilities, which in turn is a microcosm of a financial market. It behavioral primitives are conditional offers to buy or sell assets whose payoffs depend on the outcome of the game.

2 Coherent lower and upper previsions

Virtually all of the fundamental theorems of rational choice theory—subjective probability, expected utility, subjective expected utility, asset pricing, welfare economics, cardinal utilitarianism, *and* non-cooperative games—are duality theorems that can be proved by using a separating hyperplane argument. In the versions of these theorems that involve finite sets of states and/or consequences, it is a variant of Farkas' lemma, the basis of the duality theorem of linear programming:

LEMMA 1: For any matrix $\boldsymbol{G}$, either there exists a non-negative vector $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \bullet \boldsymbol{G} < \boldsymbol{0}$ or else there exists a non-negative vector $\boldsymbol{\pi}$ such that $\boldsymbol{G}\,\boldsymbol{\pi} \geq \boldsymbol{0}$, $\boldsymbol{\pi} \neq \boldsymbol{0}$.

LEMMA 2: For any matrix $\boldsymbol{G}$, either there exists a non-negative vector $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \bullet \boldsymbol{G} \leq \boldsymbol{0}$ and $[\boldsymbol{\alpha} \bullet \boldsymbol{G}]_k < 0$ or else there exists a non-negative vector $\boldsymbol{\pi}$, with $\pi_k > 0$, such that $\boldsymbol{G}\,\boldsymbol{\pi} \geq \boldsymbol{0}$.

De Finetti's (1974) "fundamental theorem of probability," as it applies to lower and upper probabilities and expectations, can be proved as follows, using the language of financial markets. Consider an agent ("she") who is uncertain about which element of a finite set $\boldsymbol{S}$ of states of the world will occur. Let N denote the number of states and let $\boldsymbol{x}$ denote an *asset*, which is an N-vector of payoffs assigned to states. The agent's *lower prevision for* $\boldsymbol{x}$ is the price $\underline{P}(\boldsymbol{x})$ that she is publicly willing to pay per unit of $\boldsymbol{x}$ in arbitrary (but small) quantities chosen by someone else. This means that for any small positive number α chosen by an observer ("he"), the agent will accept a bet whose payoff vector for her is $\alpha(\boldsymbol{x} - \underline{P}(\boldsymbol{x}))$, with the opposite payoffs to the observer.[1] For example, if N=3, $\boldsymbol{x} = (3, 1, -2)$, and $\underline{P}(\boldsymbol{x}) = 1.4$, the agent will accept a bet whose

[1] Notational conventions: Lower-case boldface letters such as $\boldsymbol{x}$ and $\boldsymbol{e}$ are used interchangeably for payoff vectors of assets and indicator vectors of events as well as for their proper names (e.g., "event $\boldsymbol{e}$" is the event whose indicator vector is $\boldsymbol{e}$). In the expression $\alpha(\boldsymbol{x} - \underline{P}(\boldsymbol{x}))$, $\boldsymbol{x}$ is a vector and α and $\underline{P}(\boldsymbol{x})$ are scalars, and the multiplication and subtraction are performed pointwise, yielding a vector whose n^{th} element is $\alpha(\boldsymbol{x}_n - \underline{P}(\boldsymbol{x}))$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors of the same length, then $\boldsymbol{xy}$ denotes their pointwise product (another vector of the same length), and $\boldsymbol{x} \bullet \boldsymbol{y}$ denotes their inner product (a scalar). If $\boldsymbol{G}$ is a matrix and $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors of appropriate length, then $\boldsymbol{x} \bullet \boldsymbol{G}$ and $\boldsymbol{G}\,\boldsymbol{y}$ denote matrix multiplication of $\boldsymbol{G}$ by $\boldsymbol{x}$ on the left or by $\boldsymbol{y}$ on the right, yielding vectors. If $\boldsymbol{\pi}$ is a probability distribution on states and $\boldsymbol{x}$ is a payoff vector and $\boldsymbol{e}$ is an indicator vector for an event, then $P_{\boldsymbol{\pi}}(\boldsymbol{x})$ is

payoff vector for her is $(1.6\alpha, -0.4\alpha, -3.4\alpha)$ for any small positive α chosen by the observer. A lower prevision for an asset may be considered as a *lower expectation*, i.e., a lower bound on its subjective expected value for the agent. In the special case where $\boldsymbol{x}$ is a binary vector that is the indicator of an event, its prevision is a *lower probability* for the event.

Lower previsions can also be assessed conditionally. If $\boldsymbol{x}$ is the payoff vector of an asset and $\boldsymbol{e}$ is the indicator vector of an event, the agent's *conditional lower prevision for* $\boldsymbol{x}$ *given* $\boldsymbol{e}$ is the price $\underline{P}(\boldsymbol{x}|\boldsymbol{e})$ that she is publicly willing to pay per unit of $\boldsymbol{x}$ in arbitrary (but small) multiples chosen by an observer, subject to the condition that the bet will be called off if $\boldsymbol{e}$ does not occur. This means that the agent will agree to accept a bet whose payoff vector for her is $\alpha(\boldsymbol{x} - \underline{P}(\boldsymbol{x}|\boldsymbol{e}))\boldsymbol{e}$, for any small positive α. To continue the previous example, if $\boldsymbol{e} = (1, 1, 0)$, which is the indicator vector for the event in which either state 1 or state 2 occurs, and $\underline{P}(\boldsymbol{x}|\boldsymbol{e}) = 2.1$, then the agent will accept a bet whose payoff vector for her is $(0.9\alpha, -1.1\alpha, 0)$. In the special case where $\underline{P}(\boldsymbol{x}|\boldsymbol{e}) = 0$, the agent is willing to pay zero for $\boldsymbol{x}$ conditional on $\boldsymbol{e}$, which means she will accept a small bet whose payoff vector is proportional to $\boldsymbol{x}$ conditional on the occurrence of $\boldsymbol{e}$. This is equivalent to an unconditional bet with payoffs proportional to $\boldsymbol{xe}$.

It remains to show that rational lower previsions satisfy the laws that ought to be satisfied by lower bounds on probabilities and expectations. Suppose that the agent assigns a conditional lower prevision $\underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m)$ to asset $\boldsymbol{x}_m$ given the occurrence of event $\boldsymbol{e}_m$, $m = 1,\ldots, M$, subject to the further requirement that bets on different events are additive, which is the way a bookmaker or financial market normally operates. For example, if the agent simultaneously assigns lower previsions $\underline{P}(\boldsymbol{x}_1|\boldsymbol{e}_1)$ and $\underline{P}(\boldsymbol{x}_2|\boldsymbol{e}_2)$ to asset $\boldsymbol{x}_1$ conditional on event $\boldsymbol{e}_1$ and asset $\boldsymbol{x}_2$ conditional on event $\boldsymbol{e}_2$, this means that for any positive real numbers α_1 and α_2 chosen by the observer, she will accept a bet whose payoff for her in state n is $\alpha_1(x_{1n} - \underline{P}(\boldsymbol{x}_1|\boldsymbol{e}_1))e_{1n} + \alpha_2(x_{2n} - \underline{P}(\boldsymbol{x}_2|\boldsymbol{e}_2))e_{2n}$, where x_{mn} and e_{mn} denote the values of $\boldsymbol{x}_m$ and $\boldsymbol{e}_m$ in state n for $m = 1, 2$.

The agent is rational *ex ante* if her previsions do not expose her to arbitrage, i.e., if the opponent is not able to make a riskless profit through a clever combination of bets. She is rational *ex post* in state k if her previsions do not allow the opponent to earn a riskless profit if state k occurs. These rationality conditions are called "coherence" and "ex post coherence," respectively. More precisely:

DEFINITION: The conditional lower previsions $\{\underline{P}(\boldsymbol{x}_1|\boldsymbol{e}_1), \ldots, \underline{P}(\boldsymbol{x}_M|\boldsymbol{e}_M)\}$ are *coherent* if there do not exist non-negative numbers $\{\alpha_1, \ldots, \alpha_M\}$ such that $\sum_{m=1}^{M} \alpha_m (x_{mn} - \underline{P}(\boldsymbol{x}_m | \boldsymbol{e}_m))e_{mn} < 0 \;\forall\, n$, i.e., the payoff to the agent is strictly negative in all states. They are *ex post coherent in state k* if a if there do not exist non-negative numbers $\{\alpha_1, \ldots, \alpha_M\}$ such that $\sum_{m=1}^{M} \alpha_m (x_{mn} - \underline{P}(\boldsymbol{x}_m | \boldsymbol{e}_m)e_{mn} \leq 0 \;\forall\, n$ with strict inequality when $n = k$, yielding a payoff to the agent that is non-positive and strictly negative in state k.

Coherence entails ex post coherence in at least one state.

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the corresponding expected value of $\boldsymbol{x}$ and $P_\pi(\boldsymbol{e})$ is the probability of $\boldsymbol{e}$, i.,e. $P_\pi(\boldsymbol{x}) = \boldsymbol{\pi} \cdot \boldsymbol{x}$ and $P_\pi(\boldsymbol{e}) = \boldsymbol{\pi} \cdot \boldsymbol{e}$. $P_\pi(\boldsymbol{x}|\boldsymbol{e})$ denotes the conditional expectation of $\boldsymbol{x}$ given the occurrence of $\boldsymbol{e}$ that is determined by $\boldsymbol{\pi}$, i.e, $P_\pi(\boldsymbol{x}|\boldsymbol{e}) = P_\pi(\boldsymbol{xe})/P_\pi(\boldsymbol{e})$ provided that $P_\pi(\boldsymbol{e}) > 0$.

THEOREM 1[2]: The conditional lower previsions $\{\underline{P}(\boldsymbol{x}_1|\boldsymbol{e}_1), \ldots, \underline{P}(\boldsymbol{x}_M|\boldsymbol{e}_M)\}$ are coherent [ex post coherent in state k] if and only if there exists a non-empty convex set Π of probability distributions on states of the world [satisfying $\pi_k > 0$] such that, for all m and all $\boldsymbol{\pi} \in \Pi$, $P_{\boldsymbol{\pi}}(\boldsymbol{x}_m|\boldsymbol{e}_m) \geq \underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m)$ or else $P_{\boldsymbol{\pi}}(\boldsymbol{e}_m) = 0$.

Proof: Let $\boldsymbol{G}$ denote the matrix whose m^{th} row is the vector $(\boldsymbol{x}_m - \underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m))\boldsymbol{e}_m$ of payoffs to the agent for the conditional bet determined by the assignment of prevision $\underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m)$ to asset $\boldsymbol{x}_m$ conditional on event $\boldsymbol{e}_m$. The conditional lower previsions $\{\underline{P}(\boldsymbol{x}_1|\boldsymbol{e}_1),\ldots, \underline{P}(\boldsymbol{x}_M|\boldsymbol{e}_M)\}$ are coherent if and only if there does not a exist non-negative vector $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \bullet \boldsymbol{G} < \boldsymbol{0}$. By Lemma 1, this is true if and only there exists a non-negative vector $\boldsymbol{\pi}$ such that $\boldsymbol{G}\ \boldsymbol{\pi} \geq \boldsymbol{0}$, $\boldsymbol{\pi} \neq \boldsymbol{0}$, which can be normalized so that its elements sum to 1, a probability distribution. The condition $\boldsymbol{G}\ \boldsymbol{\pi} \geq \boldsymbol{0}$ means $P_{\boldsymbol{\pi}}((\boldsymbol{x}_m - \underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m))\boldsymbol{e}_m) \geq 0$, or equivalently $P_{\boldsymbol{\pi}}(\boldsymbol{x}_m\boldsymbol{e}_m) \geq \underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m))P_{\boldsymbol{\pi}}(\boldsymbol{e}_m)$, for all m. This is trivially true if $P_{\boldsymbol{\pi}}(\boldsymbol{e}_m) = 0$, because both sides are zero. If $P_{\boldsymbol{\pi}}(\boldsymbol{e}_m) > 0$, it rearranges to $P_{\boldsymbol{\pi}}(\boldsymbol{x}_m\boldsymbol{e}_m)/P_{\boldsymbol{\pi}}(\boldsymbol{e}_m) \geq \underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m)$, which by definition means $P_{\boldsymbol{\pi}}(\boldsymbol{x}_m|\boldsymbol{e}_m) \geq \underline{P}(\boldsymbol{x}_m|\boldsymbol{e}_m)$. The corresponding result for ex post coherence in state k follows by applying Lemma 2 in place of Lemma 1. ■

Coherent lower previsions therefore have the properties of lower probabilities and expectations determined by a convex set of probability distributions, which can be interpreted to represent the possibly-imprecise beliefs of the agent, if she has linear utility for money.

An under-appreciated property of de Finetti's operational definition of subjective probabilities and expectations is that it does not merely define them: it makes them common knowledge in the everyday specular sense of the term. The prices are visible to both actors in the scene, and the actors both know it, and both know that they both know it, and so on, and the meaning of the numbers is commonly understood by virtue of the opportunities that they create for reciprocal financial transactions. This is a property of posted prices in general. They do not only simplify the decision-making of consumers and investors: they are also credible and commonly known numerical measurements of the comparative beliefs and values of those who post them.

It might be argued that game-theoretic techniques should be used to address the question of why and how the agent should offer distinct lower and upper previsions (bid and ask prices) in her interaction with the observer, or whether she should offer to bet at all. There might be asymmetric information or incentives for secrecy or deception or speculation that would motivate the agent to set her bid prices for assets at levels other than her true lower bounds on their expected payoffs, whatever "true" might mean. This would merely beg the question of how the rules of the higher-order game would come to be commonly known in numerical terms. If an infinite regress is to be avoided, then at some level of description the amount of private information about her beliefs and values that an agent is willing to publicly reveal is a behavioral primitive. In the sequel, the game-theoretic argument will be turned on its head: the fundamental theorem of non-cooperative games is merely an extension of the fundamental theorem of probability to multiple actors in the same scene.

[2] De Finetti (1937) separately derives the additive and multiplicative laws of probability from the requirement of coherence by evaluating the determinants of matrices whose rows are the payoff vectors of acceptable bets. A proof of the more general statement of the theorem via a separating hyperplane method is given by Smith (1961). The "ex post" variant of the theorem is discussed by Nau (1995).

3 Previsions conditioned on one's own moves

In the assessment of previsions via offers to bet, there is no requirement that states of the world should be events that are beyond the agent's control. However, an observer might be reluctant to take the other side of any bet whose payoff depends on an event that they both know the agent *does* control, and conversely the agent might be reluctant to offer to bet on events that she knows to be controlled by others. An important special case is one in which the state space can be partitioned as $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, where $\mathcal{S}_1$ is a set of events that the agent controls (her own moves) while $\mathcal{S}_2$ is a set of events outside her control (moves of nature or other agents). If $\boldsymbol{e}$ is an event that is measurable with respect to $\mathcal{S}_1$ (the indicator for a move or subset of moves of the agent), and $\boldsymbol{x}$ is the payoff vector of an asset that is measurable with respect to $\mathcal{S}_2$ (a bet whose payoff depends only on moves of others), it might be reasonable for the agent to assert a lower prevision for $\boldsymbol{x}$ conditional on $\boldsymbol{e}$. If she asserts that $\underline{P}(\boldsymbol{x}|\boldsymbol{e}) = 0$, it means that she will accept a small bet whose payoff vector is proportional to $\boldsymbol{x}$ under the same conditions in which she would choose the move $\boldsymbol{e}$, or equivalently, she will accept a small bet whose payoff vector is proportional to $\boldsymbol{xe}$. Such a bet reveals some information about the agent's payoff function in the game she is playing against nature or her adversaries without necessarily revealing the move she intends to make. Namely, her payoffs in the game are such that her best move is $\boldsymbol{e}$ only under conditions where her prevision for $\boldsymbol{x}$ is non-negative. This method for revealing limited information about one's payoff function yields enough detail about the rules of a non-cooperative game to determine its equilibria, as will be shown next.

4 Imprecise equilibria of games

Let $\mathcal{G}$ denote a non-cooperative game among I players, each having a finite set of strategies. Let $\mathcal{S} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_I$ denote the set of outcomes, where $\mathcal{S}_i$ is the set of index numbers for strategies of player i. Let $\boldsymbol{s} = (s_1, \ldots, s_I)$ denote a generic outcome, in which s_i is the strategy chosen by player i. Let $\boldsymbol{x}_i$ denote the payoff function (an $|\mathcal{S}|$-dimensional vector) for player i, whose value in outcome $\boldsymbol{s}$ is $x_i(\boldsymbol{s})$. Assume that payoffs are measured in units of money and that the players are risk neutral. (The risk neutrality assumption will be relaxed later.) The "true" game $\mathcal{G}$ is therefore defined by the sets of strategies $\{\mathcal{S}_i\}$ and payoff vectors $\{\boldsymbol{x}_i\}$.

Let $\boldsymbol{e}_{ij}$ denote the event in which player i plays her j^{th} strategy, and for every $j \in \mathcal{S}_i$, let $\boldsymbol{x}_{ij}$ denote a vector of payoffs that is obtained from $\boldsymbol{x}_i$ as follows: $\boldsymbol{x}_{ij}(\boldsymbol{s}) = \boldsymbol{x}_i(s_1, \ldots, j, \ldots, s_N)$, where the j occurs in the i^{th} position. In other words, $\boldsymbol{x}_{ij}(\boldsymbol{s})$ is the profile of payoffs that player i receives by playing her j^{th} strategy while all other players play according to $\boldsymbol{s}$. Note that there is some duplication of information in the structure of $\boldsymbol{x}_{ij}(\boldsymbol{s})$: it contains multiple copies of the payoff profile that player i obtains by playing j, because the element of $\boldsymbol{x}_{ij}(\boldsymbol{s})$ in coordinate $(s_1, \ldots, s_i, \ldots, s_N)$ is the same for all values of s_i.

Suppose that the payoff functions $\{\boldsymbol{x}_i\}$ are not commonly known *a priori* and must therefore be revealed through some credible language of communication. The language that will be used here is the same one that was sketched in the previous section. To see how it works in the game, observe that in the event that player i chooses her j^{th} strategy, she must weakly prefer the profile of payoffs she gets by playing strategy j to the profile of payoffs she would have gotten by playing any other strategy k. In the terms introduced above, she evidently prefers $\boldsymbol{x}_{ij}$ over $\boldsymbol{x}_{ik}$ in the event that $\boldsymbol{e}_{ij}$ occurs, which means that she would trade $\boldsymbol{x}_{ik}$ for $\boldsymbol{x}_{ij}$ conditional on $\boldsymbol{e}_{ij}$. Such a trade is equivalent to an unconditional bet with a payoff vector of $(\boldsymbol{x}_{ij} - \boldsymbol{x}_{ik})\boldsymbol{e}_{ij}$. If the agent wants to let this information about her payoff function be common knowledge, she can publicly offer to accept a small bet whose payoff vector is proportional to $(\boldsymbol{x}_{ij} - \boldsymbol{x}_{ik})\boldsymbol{e}_{ij}$ at the discretion of an observer. Or, to turn the story around, if by magic her payoff function $\boldsymbol{x}_i$ is already common knowledge, then it is also common knowledge that she will accept such a bet.[3] Note that she is not betting directly on her own strategy. Rather, her own strategy is used as a conditioning event for bets on what other players will do. Bets that are conditioned on the player's own strategy, which may be uncertain to the observer and the other players, do not necessarily reveal her actual state of information or her intended move.

Suppose that all the players offer to accept small conditional bets that are determined by their true payoff functions in the manner described above. Let $\boldsymbol{G}$ denote the matrix whose columns are indexed by outcomes of the game, whose rows are indexed by ijk, and whose ijk^{th} row is $(\boldsymbol{x}_{ij} - \boldsymbol{x}_{ik})\boldsymbol{e}_{ij}$, the payoff vector of the bet that is acceptable to player i in the event that she chooses strategy j in preference to strategy k. Then, under the assumption that such bets may be non-negatively linearly combined, an observer of the game may choose a non-negative vector of multipliers $\boldsymbol{\alpha}$ to construct an acceptable bet that yields a total payoff vector of $\boldsymbol{\alpha} \bullet \boldsymbol{G}$ to the players and the opposite payoffs to himself.

$\boldsymbol{G}$ will be henceforth called the "revealed rules of the game matrix" because, as will be shown, it contains all the commonly-knowable information about the rules that is actually used in determining the equilibria of non-cooperative games. However, $\boldsymbol{G}$ does not contain all the information about the true game $\mathcal{G}$ that is economically important to the players. In particular, it does not reveal the benefits that a given player might obtain from changes in the strategies of the other players, holding her own strategy fixed. The latter information is subtracted out when the calculation $(\boldsymbol{x}_{ij} - \boldsymbol{x}_{ik})\boldsymbol{e}_{ij}$ is performed. All that remains is information about how a given player would benefit by changing her own strategy, holding the strategies of the *other* players fixed. This is the essence of "non-cooperative" game-playing. The players do not consider the implications of their own play for the payoffs of other players, nor do they expect the other players to show that consideration to them.

Under the assumptions given above, we can define what it means for the game to be played rationally by applying the concept of ex post coherence jointly to all the players. Consider an observer who knows nothing about the game except the bets that the players have offered, which is the minimal information about the game's rules that is common knowledge. Suppose that he does not want to speculate on the game's outcome, but he would like to make a riskless profit if possible. From the observer's perspective, if several bets are placed on the same table at the same time, it doesn't matter whether they are offered by one individual or by many who are all looking each other in the eye. If the observer manages to pick their pockets, the players have behaved irrationally.

[3] Strictly speaking, the *choice* of strategy j in the presence of k can only be interpreted to mean a *preference* for j over k if the agent has complete preferences, requiring precise beliefs. Here, offers to bet are assumed to occur at a point in time when the agents may not yet have formed precise beliefs about what their opponents will do, but they expect that they will have done so by the time they are called upon to move. In the meantime they are making assertions about constraints that precise beliefs would have to satisfy in order for them to prefer one strategy over another, thereby partially revealing their payoff functions.

DEFINITION: The strategy $\boldsymbol{s}$ is *jointly coherent* if there does *not* exist a non-negative $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \bullet \boldsymbol{G} \leq \mathbf{0}$ and $[\boldsymbol{\alpha} \bullet \boldsymbol{G}](\boldsymbol{s}) < 0$, i.e., if, under the revealed rules of the game, there is no system of system of bets under which the observer cannot lose and will win a positive amount from the players if they play $\boldsymbol{s}$.

Fortunately for the players, there is always at least one jointly coherent strategy: they are not doomed to exploitation if they honestly reveal some information about their payoff functions.[4] The interesting question is whether there are strategies that are *not* jointly coherent, and if so, how are they characterized.

In general, the players might choose either pure or randomized strategies, and randomized strategies might be either independent or correlated. Correlated randomization of strategies could be carried out with the help of a mediator but does not necessarily require it: flipping a coin or playing paper-scissors-rock are familiar correlation devices that do not require a mediator, and a taking-turns convention in repeated play could be viewed as a correlation device from the perspective of an observer who doesn't who whose turn it is. Let $\boldsymbol{\pi}$ denote a (possibly-degenerate) probability distribution over the outcomes of the game, and suppose that the players *do* employ a mediator who is instructed to randomly draw a joint strategy $\boldsymbol{s}$ according to the distribution $\boldsymbol{\pi}$ and then privately recommend to each player that she should play her own part of it. Thus, player i hears only her own recommended strategy, s_i, not those of the other players. Under these conditions, $\boldsymbol{\pi}$ is a common prior distribution over recommended joint strategies in the game, and each player can use Bayesian updating to compute a posterior distribution for the recommendations that were received by the other players, given her own recommendation. If each player's recommended strategy is optimal for her *a posteriori* when the others play their own recommended strategies, then $\boldsymbol{\pi}$ is a correlated equilibrium of the game (Aumann 1974, 1987). More precisely:

DEFINITION: $\boldsymbol{\pi}$ is a *correlated equilibrium* of $\mathcal{G}$ if and only if $\boldsymbol{G}\,\boldsymbol{\pi} \geq \mathbf{0}$, which means that for every player i and every recommended strategy j and alternative strategy k of that player, either $P_{\pi}(\boldsymbol{e}_{ij}) = 0$ (the probability of strategy j being recommended to player i is zero) or else $P_{\pi}(\boldsymbol{x}_{ij}(\boldsymbol{s}) - \boldsymbol{x}_{ik}(\boldsymbol{s})|\boldsymbol{e}_{ij}) \geq 0$ (the conditional expected payoff of strategy j is greater than or equal to the conditional expected payoff of strategy k when j is recommended).

Because the set of all correlated equilibria of $\mathcal{G}$ is determined by a system of linear inequalities, it is a convex polytope—a tractable geometrical object—which will henceforth be denoted by $\Pi_{\mathcal{G}}$. A *Nash equilibrium* is a special case of a correlated equilibrium in which $\boldsymbol{\pi}$ is independent between players, allowing each player to perform her own randomization (if necessary) without a mediator. The set of Nash equilibria is not necessarily convex or connected or bounded by points with rational coordinates, and it can be rather difficult to compute, particularly in games with more than 2 players.

[4] A proof of this result is given in Nau and McCardle (1990). A proof of the dual condition, which (by Theorem 2) is the existence of a correlated equilibrium, is given by Hart and Schmeidler (1989). These proofs are more elementary than the proof of existence of a Nash equilibrium insofar as they do not invoke a fixed-point theorem. In Nau and McCardle's proof, the result follows from the existence of a stationary distribution of a Markov chain.

In these terms we can prove a "fundamental theory of non-cooperative games" which is the strategic generalization of the fundamental theorem of probability. Actually, the theorem and its proof are merely a restatement of the fundamental theorem of probability and *its* proof for the special case in which conditional previsions are jointly announced by two or more individuals and the assets and conditioning events to which they refer have a special structure that is determined by a non-cooperative game they are playing.

THEOREM 2 (Nau and McCardle 1990): In a game among risk neutral players, a strategy is jointly coherent if and only if there exists a correlated equilibrium in which it has positive probability.

Proof: By Lemma 2, either there exists a non-negative vector $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \bullet \boldsymbol{G} \leq \mathbf{0}$ and $[\boldsymbol{\alpha} \bullet \boldsymbol{G}](\boldsymbol{s}) < 0$ or else there exists a non-negative vector $\boldsymbol{\pi}$, with $\pi(\boldsymbol{s}) > 0$, such that $\boldsymbol{G}\,\boldsymbol{\pi} \geq \mathbf{0}$. ■

Hence, the players are rational ex post if and only if they behave as if they had implemented a correlated equilibrium, i.e., if they play a strategy that could have occurred with positive probability in such an equilibrium.[5] But even more can be said: lower and upper bounds can be placed on the players' jointly-held previsions for outcomes of the game and any side bets that might be placed on it, namely the bounds that are determined by the convex polytope $\Pi_{\mathcal{G}}$ of correlated equilibria. On this basis it is appropriate to consider $\Pi_{\mathcal{G}}$ to be the rational "solution" of the game when it is played non-cooperatively in the absence of any constraints other than coherence, and in general it is a solution in terms of imprecise probabilities.[6]

There are three important special cases of a 2x2 game. In the first of them both players have strictly dominant pure strategies and the generic payoff matrix can be written as

	L	R
T	$a, d-1$	c, d
B	0, 0	$c-1, b$

Table 1

where a and b are both positive. (T, B, L, R stand for Top, Bottom, Left, Right.) This choice of parameters is without loss of generality because each player's payoff function has only two degrees of freedom apart from origin and scale, and it has been chosen because a and b (only) determine the noncooperative equilibria of the game, while c and d are externalities that determine the benefits of cooperation (if any). Row's payoff is increased by the amount c if Column's strategy is changed from L to R, ceteris paribus, and Column's payoff is increased by the amount d if Row's strategy is changed from B to T.

[5] In games of incomplete information, joint coherence leads to a correlated generalization of Bayesian equilibrium (Nau 1992).

[6] This approach can be generalized to the situation in which players do not exactly know their own payoffs. If each payoff in the game matrix is known by its recipient only to lie within some interval, then the ijk^{th} row of $\boldsymbol{G}$ becomes $(\boldsymbol{x}_{ij}^{max} - \boldsymbol{x}_{ik}^{min})\boldsymbol{e}_{ij}$, where $\boldsymbol{x}_{ij}^{max}$ and $\boldsymbol{x}_{ik}^{min}$ are pointwise maxima and minima of the possible payoffs of strategies j and k for player i. This means that in the event that player i chooses strategy j over strategy k, the minimal requirement that her conditional beliefs must satisfy is that her best possible lower prevision for the payoff of j should be at least as great as her worst possible lower prevision for the payoff of k. In general, this sort of payoff-imprecision weakens the constraints and therefore enlarges the set of correlated equilibria.

In the game of Table 1, T is strictly dominant for Row and R is strictly dominant for Column, so the unique strategically rational solution is TR. Whether this solution is socially rational depends on *c* and *d*. For example, if $a = b = 1$ and $c = d = -10$, the game is

	L	R
T	1, –11	–10, –10
B	0, 0	–11, 1

Table 2

This is a Prisoner's Dilemma in which TR is Pareto inefficient, being strongly dominated by BL. However, if $a = b = 1$ and $c = d = +10$, the game becomes

	L	R
T	1, 9	10, 10
B	0, 0	9, 1

Table 3

which is a win-win situation: the strategically rational solution is also socially rational. These two games are equivalent from the asocial perspective of noncooperative game theory, whose solution concepts disregard the externalities *c* and *d*. Those parameters are not really a part of the rules according to which the game is played, which is the source of the dilemma in Table 2.

The second important special case is the one in which neither player has a weakly or strongly dominated strategy and their interests are strictly opposed in the sense that at least one player would want to unilaterally defect from any pure strategy solution that might be proposed. Such a game has a unique equilibrium in independently randomized strategies. The generic payoff matrix can be written as

	L	R
T	$a, d-1$	c, d
B	0, 0	$c+1, -b$

Table 4

where *a* and *b* are both positive. (This differs from the game matrix of Table 1 in that -1 is changed to +1 and *b* is changed to $-b$ in cell BR.) TL cannot be an equilibrium of this game because Column would want to switch from L to R if Row played T, and Row would then want to switch from T to B, and Column would then want to switch from R to L, and Row would then want to switch from B to T, forming a cycle. In the unique equilibrium, Row plays T with probability $b/(1+b)$ and Column plays R with probability $a/(1+a)$. For example, if $a = 2$, $b = 3$ and $c = d = 0$, the payoff matrix is

	L	R
T	2, –1	0, 0
B	0, 0	1, –3

Table 5

The unique solution is for Row to play T with probability 3/4 and for Column to play R with probability 2/3. The outcome with the highest probability (1/2) is TR and one with the lowest probability (1/12) is BL. The same solution would be obtained for any other values of c and d. For example, with $c = d = -10$, we get the strategically equivalent game

	L	R
T	2, –11	–10, –10
B	0, 0	–9, –3

Table 6

in which the players strongly prefer the lowest-probability outcome over the highest-probability one. As in the Prisoner's Dilemma, strategic rationality forces the players to behave in a way that assigns high probability to an unattractive outcome if externalities are not aligned with strategic incentives.

The third, and most interesting, special case is the one in which neither player has a weakly or strongly dominated strategy and their interests are not strictly opposed: a so-called coordination game. The generic payoff matrix of such a game can be written as

	L	R
T	a, d+1	c, d
B	0, 0	c+1, b

Table 7

where (again) a and b are both positive. . (This differs from the payoff matrix of Table 1 in that –1 is changed to +1 in both cells TL and BR.) Games of this form include Battle-of-the-Sexes and Chicken. For example, if $a = b = 2$ and $c = d = 0$, the payoff matrix is

	L	R
T	2, 1	0, 0
B	0, 0	1, 2

Table 8

which is a version of Battle-of-the-Sexes. Here the players both prefer TL or BR to either of the other alternatives, and between these Row has a strict preference for TL and Column has a strict preference for BR. If $a = b = 1$ and $c = d = 2$, the payoff matrix is

	L	R
T	1, 3	2, 2
B	0, 0	3, 1

Table 9

which is a version of Chicken, in which only BL is strictly dominated.

For the coordination game of Table 7, the rules-of-the-game matrix, $\boldsymbol{G}$, is

	TL	TR	BL	BR
1TB	a	-1	0	0
1BT	0	0	$-a$	1
2LR	1	0	$-b$	0
2RL	0	-1	0	b

Table 10

which (as always) does not involve the externalities c and d. Here the row label 1TB means G_{1TB}, the payoff vector of the bet that player 1 is willing to accept when she chooses Top in preference to Bottom, etc.

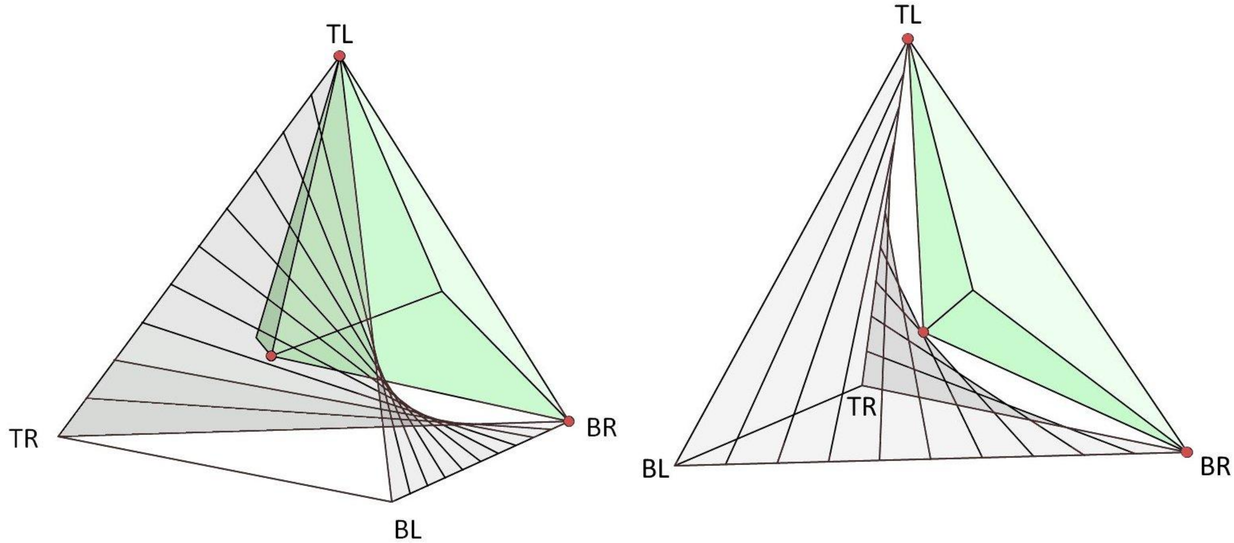
The correlated equilibrium polytope of this game is a hexahedron with five vertices, of which two are pure Nash equilibria, one is a completely mixed Nash equilibrium, and the other two are non-Nash correlated equilibria. The probabilities assigned to outcomes of the game in each of these extremal equilibria are:

	TL	TR	BL	BR	Nash?
Vertex 1	1	0	0	0	Yes
Vertex 2	0	0	0	1	Yes
Vertex 3	$b/(a+ab+1+b)$	$ab/(a+ab+1+b)$	$1/(a+ab+1+b)$	$a/(a+ab+1+b)$	Yes
Vertex 4	$b/(a+ab+b)$	$ab/(a+ab+b)$	0	$a/(a+ab+b)$	No
Vertex 5	$b/(a+1+b)$	0	$1/(a+1+b)$	$a/(a+1+b)$	No

Table 11

In the mixed Nash equilibrium (vertex 3), the probabilities of the four game outcomes are in the proportions a:ab:1:b, which means that Row chooses T with probability $a/(1+a)$ and Column independently chooses R with probability $b/(1+b)$. The two non-Nash correlated equilibria (vertices 4 and 5) are obtained from the mixed Nash equilibrium by zeroing-out the probability on one of the two outcomes TR or BL and then renormalizing. From the structure of $\boldsymbol{G}$ it is obvious that this tactic will yield extremal correlated equilibria, because the columns that are associated with TR and BL contain only non-positive values, hence zeroing-out their probabilities can only loosen the incentive constraints.

Two views of the geometry of the correlated equilibrium polytope are shown below for the case a=b=2. The simplex of all probability distributions on outcomes of the game is a tetrahedron, the set of distributions that are independent between players is a saddle, the correlated equilibrium polytope is a hexahedron, and their 3 points of intersection are the Nash equilibria. Nash equilibria always lie on the surface of the correlated equilibrium polytope, but in larger games they need not be vertices of it (Nau et al. 2004).



Externalities play no role in determining the shape of the correlated equilibrium polytope, but they do determine which of its vertices are Pareto efficient.

THEOREM 3: In the polytope of correlated equilibria of the coordination game of Table 7:

(i) At least one of the two pure Nash equilibria is weakly Pareto efficient, and if only one of them is weakly Pareto efficient, then it is the only correlated equilibrium that is weakly Pareto efficient.

(ii) The mixed Nash equilibrium is never weakly Pareto efficient.

(iii) The extremal correlated equilibrium that places zero probability on BL is strongly [weakly] Pareto efficient if only if, for all $\lambda \in (0,1)$, both [at least one of] the conditions $c \geq a + (1-\lambda)/\lambda$ and $d \geq b + \lambda/(1-\lambda)$ are satisfied.[7]

(iv) The extremal correlated equilibrium that places zero probability on TR is strongly [weakly] Pareto efficient if only if, for all λ, both [at least one of] the conditions $c \leq -1 - \lambda/(1-\lambda)a$ and $d \leq -1 - ((1-\lambda)/\lambda)b$ are satisfied.

The proof is given in the appendix. The implication of the Theorem is that the set of efficient correlated equilibria of this game consists of one of the following: (i) one of the two pure Nash equilibria; (ii) both of the two pure Nash equilibria and the line segment connecting them; (iii) the face of the polytope determined by the two pure Nash equilibria and one of the extremal non-Nash correlated equilibria.

[7] These are the necessary and sufficient conditions under which this particular correlated equilibrium is not weakly or strongly dominated by some convex combination of the two pure Nash equilibria, and similarly for (iv).

5 Risk aversion & risk neutral probabilities

The results of the previous sections require the players to be risk neutral, i.e., to have state-independent linear utility for money. The more general case of risk averse players will be considered next, and it will be shown that risk aversion leads them to hedge their bets, making the revealed set of equilibria larger than it would have been otherwise. Furthermore, when players are risk averse, side bets may provide opportunities for Pareto-improving modifications of the rules of the game, which leads to some blurring of the distinction between strategic and competitive equilibria. In extreme cases, players may be able to hedge their positions so as to decouple their payoff functions and exit from the game altogether. To set the stage, some general remarks on the modeling of risk aversion are appropriate.

If an agent is risk averse rather than risk neutral, and if she has substantial prior stakes in events ("background risk"), then Theorem 1 still holds, but its parameters have a different interpretation. Suppose that the agent has subjective expected utility preferences and her risk attitude is represented by a strictly concave von Neumann-Morgenstern utility function $U(x)$, with its derivative denoted by $U'(x)$, and suppose that her background risk is represented by a payoff vector $\boldsymbol{z}$ whose elements differ across states by amounts that are large enough to cause substantial variations in the marginal utility of money. Then her acceptance of an additional small bet $\boldsymbol{x}$ will not be based on its expected value but rather on its expected marginal utility in the context of $\boldsymbol{z}$. If the agent's beliefs are represented by a precise probability distribution $\boldsymbol{p}$, then her status quo expected utility is $E_{\boldsymbol{p}}[U(\boldsymbol{z})]$. A bet $\boldsymbol{x}$ will be acceptable to her if it maintains or increases her expected utility, i.e., if $E_{\boldsymbol{p}}[U(\boldsymbol{z}+\boldsymbol{x})] - E_{\boldsymbol{p}}[U(\boldsymbol{z})] \geq 0$.

If the elements of $\boldsymbol{x}$ are small enough in magnitude so that only first-order effects are important, then $\boldsymbol{x}$ is acceptable if $E_{\boldsymbol{p}}[U'(\boldsymbol{z})\boldsymbol{x}] \geq 0$, or equivalently if $E_{\pi}[\boldsymbol{x}] \geq 0$, where $\boldsymbol{\pi}$ is a probability distribution obtained by multiplying the true probability distribution $\boldsymbol{p}$ pointwise by the marginal utility vector $U'(\boldsymbol{z})$ and then re-normalizing, so that $\pi(\boldsymbol{s}) \propto p(\boldsymbol{s})U'(z(\boldsymbol{s}))$. This is the *risk neutral probability distribution of the agent* at $\boldsymbol{z}$, because she evaluates small bets in a seemingly risk neutral way using $\boldsymbol{\pi}$ rather than her true subjective probability distribution $\boldsymbol{p}$. The risk neutral distribution of the agent is not uniquely determined by beliefs: it also depends on her background risk and her attitude toward it.[8]

In a financial market, the necessary and sufficient condition for asset prices to create no arbitrage opportunities is that there should exist a probability distribution under which every asset's expected payoff (discounted at the risk-free rate of interest if time is a factor), lies between its bid and ask prices. This result is known as the "fundamental theorem of asset pricing," and it is merely de Finetti's fundamental theorem of probability applied to asset prices offered by the whole market rather than by a single individual. The probability distribution that prices the assets is called the *risk neutral probability distribution of the market*, because it prices them in a seemingly risk neutral way, and it can be determined from prices of options or Arrow securities.[9] Because of friction and incompleteness, the market's risk neutral distribution is usually not

[8] The role of risk neutral probabilities in modeling a single agent's aversion to risk—and also ambiguity—is discussed in more detail by Nau (2001, 2003, 2011).

[9] The literature on arbitrrage pricing and risk neutral probabilities in finance traces back to the seminal work of Black and Scholes, Merton, Cox, Ross, Rubinstein, and many others in the 1970's, although the connection with de Finetti's use of the no-arbitrage principle in subjective probability, dating to the 1930's, was not noticed until later.

unique. Rather, there is a convex set of risk neutral distributions determined by bid and ask prices for assets.

In equilibrium, the marginal prices that agents are willing to pay for financial assets must agree with market prices, which means that the risk neutral probability distributions of all the agents must agree with the risk neutral probability distribution of the market. More precisely, the set of risk neutral distributions that is determined by bid and ask prices in the market is the intersection of all the sets of risk neutral distributions that are determined by bid and ask prices of individuals, which is non-empty if and only if there are no arbitrage opportunities. Thus, rational behavior in markets requires the agents to "agree" on risk neutral probabilities to the extent that their sets of personal risk neutral probabilities must have a least one point in common. In the special case where the agents have complete preferences and the market is also complete and frictionless, the risk neutral probabilities of the agents and the market are uniquely determined and must be identical.

6 Risk neutral equilibria

When agents are risk averse with significant prior stakes in events, their lower and upper previsions determined by offers to accept small bets must be interpreted as lower and upper expectations with respect to convex sets of risk neutral probabilities, rather than true subjective probabilities, as discussed above. The same consideration applies to the analysis of games. A game's own payoffs are a source of background risk with respect to bets on its outcome, and if the players are sufficiently risk averse, this will give rise to distortions when the rules of the game are revealed through betting. The result will be that a rational solution of the game is characterized by a convex set of equilibria whose parameters are risk neutral probabilities.

Suppose that each player has strictly risk averse subjective-expected-utility preferences with respect to profiles of monetary payoffs in the game, and let U_i denote the strictly concave von Neumann-Morgenstern utility function of player i. Then the payoff profiles $\{x_i(\boldsymbol{s})\}$ translate into utility profiles $\{U_i(x_i(\boldsymbol{s}))\}$. Let $\mathcal{G}^*$ denote the "true" game that is determined by the utility profiles. If U_i' denotes the first derivative of U_i, strict concavity requires that $U_i'(x) < U_i'(y)$ whenever $x > y$. Let $\boldsymbol{u}_i$ denote the utility payoff vector for player i, whose value in outcome $\boldsymbol{s}$ is $U_i(x_i(\boldsymbol{s}))$, and let $\boldsymbol{u}_i'$ denote the corresponding marginal utility vector whose value in outcome $\boldsymbol{s}$ is $U_i'(x_i(\boldsymbol{s}))$. Also, let $\boldsymbol{u}_{ij}$ denote the vector constructed from $\boldsymbol{u}_i$ in the same way that $\boldsymbol{x}_{ij}$ was constructed from $\boldsymbol{x}_i$, namely $u_{ij}(\boldsymbol{s}) = U_i(x_{ij}(\boldsymbol{s}))$. In other words, $u_{ij}(\boldsymbol{s})$ is the utility that player i would receive by playing her j^{th} strategy when all others play according to $\boldsymbol{s}$. Let $\boldsymbol{u}_{ij}'$ denote the corresponding profile of marginal utilities for money, i.e., $u_{ij}'(\boldsymbol{s}) = U_i'(x_{ij}(\boldsymbol{s}))$. As in the case of $\boldsymbol{x}_{ij}$, there is some duplication of information insofar as $u_{ij}(\boldsymbol{s})$ and $u_{ij}'(\boldsymbol{s})$ do not depend on the value of s_i.

By an argument analogous to the one used in the risk neutral case, player i will choose strategy j in preference to strategy k only if her beliefs are such that she would be willing to exchange the utility profile $\boldsymbol{u}_{ik}$, for the utility profile $\boldsymbol{u}_{ij}$, hence a small monetary bet yielding a profile of changes in *marginal* utility that is proportional to $\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik}$ should be acceptable if the event $\boldsymbol{e}_{ij}$ is observed to occur. When strategy j is chosen, the agent's profile of marginal utilities for money is $\boldsymbol{u}_{ij}'$, and a monetary bet that yields a profile of marginal utilities proportional to $\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik}$ can be obtained by dividing the utilities by the corresponding marginal utilities. Thus, agent i should be willing to accept a small bet whose monetary payoffs are proportional to $(\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}'$ conditional on the occurrence of $\boldsymbol{e}_{ij}$. Such a bet has an unconditional payoff vector of $((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}')\boldsymbol{e}_{ij}$ in units of money.

Let $\boldsymbol{G}^*$ now denote the matrix whose rows are indexed by ijk and whose columns are indexed by $\boldsymbol{s}$ and whose ijk^{th} row is the vector $((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}')\boldsymbol{e}_{ij}$ defined above. This is the revealed rules matrix for the game $\mathcal{G}^*$, representing the information about the game that can be made common knowledge through unilateral offers to accept small bets when the players are risk averse. If an observer chooses a small non-negative vector $\boldsymbol{\alpha}$ of multipliers for these bets, the players as a group will receive the vector of payoffs $\boldsymbol{\alpha} \bullet \boldsymbol{G}^*$ and the observer will receive the opposite payoffs. The same rationality criterion that was applied in the risk neutral case also applies here: an outcome $\boldsymbol{s}$ is jointly coherent if there is no non-negative $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \bullet \boldsymbol{G}^* \leq \boldsymbol{0}$ and $[\boldsymbol{\alpha} \bullet \boldsymbol{G}^*](\boldsymbol{s}) < 0$.[10] The definition of correlated equilibrium and the fundamental theorem of games can now be generalized accordingly. The proof is the same.

DEFINITION: $\boldsymbol{\pi}$ is a *risk neutral equilibrium* of $\mathcal{G}^*$ if and only if $\boldsymbol{G}^*\boldsymbol{\pi} \geq \boldsymbol{0}$, which means that for every player i and every strategy j and alternative strategy k of that player, either $P_{\boldsymbol{\pi}}(\boldsymbol{e}_{ij}) = 0$ or else $P_{\boldsymbol{\pi}}((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}')|\boldsymbol{e}_{ij}) \geq 0$.

THEOREM 4: In a game among risk averse players, a strategy is jointly coherent if and only if there is a risk neutral equilibrium in which it has positive probability.

To provide a story to go with this solution concept, suppose that the players employ a mediator who will use a possibly-correlated randomization device to recommend strategies to them privately, but in this more general case they do not necessarily agree on the true prior probabilities of the outputs of the device. For example, the device may take some of its input data from financial markets or from political or sporting or weather events. Suppose that through side bets with each other or through participation in a public betting market for the input events, they have arrived at a common prior *risk neutral* probability distribution $\boldsymbol{\pi}$ for the outputs of the device. Finally, suppose they will not have the opportunity to directly observe any of the input or output data prior to making their moves *except* for the private recommendations they receive from the mediator, who *will* have observed the data. Under these conditions, for all i, j, and k, the constraint $P_{\boldsymbol{\pi}}((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}')|\boldsymbol{e}_{ij}) \geq 0$ implies $\boldsymbol{p}_{ij} \bullet (\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik}) \geq 0$, which means that according to player i's own private beliefs, strategy j yields an expected utility greater than or equal to that of the alternative strategy k when j is recommended to her, so it is optimal for each player to follow the mediator if all others do, and this is common knowledge. Hence a game

[10] When the utility functions of the players are strictly concave rather than linear, the bet with payoff vector $((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}')\boldsymbol{e}_{ij}$ is technically only "marginally" acceptable to player i, so a bet with an aggregate payoff vector of $\boldsymbol{\alpha} \bullet \boldsymbol{G}^*$ may not be quite acceptable to the players for finite $\boldsymbol{\alpha}$. In such a case the observer may need to make a small side payment to the players to get them to agree to the deal, which makes the observer's position not entirely riskless. However, if $\boldsymbol{\alpha} \bullet \boldsymbol{G}^* \leq \boldsymbol{0}$ and $[\boldsymbol{\alpha} \bullet \boldsymbol{G}^*](\boldsymbol{s}) < 0$, then by choosing $\boldsymbol{\alpha}$ sufficiently small, the magnitude of the required side payment can be made arbitrarily small in relative terms in comparison to the aggregate loss the players will suffer if they play $\boldsymbol{s}$, which will be considered here as sufficient grounds for not playing $\boldsymbol{s}$. This could be made precise by using the concept of ε-acceptable bets (Nau 1995), but it will not be pursued here in the interest of brevity.

among risk averse players is played coherently if and only if it is played "as if" with the help of a mediator who uses an incentive-compatible device with respect to whose outputs the players have a common prior risk neutral distribution, although their unobserved true distributions may differ.

A risk neutral equilibrium is a special case of a *subjective correlated equilibrium* (Aumann 1974, 1987), which is an equilibrium that can be implemented through a mediator who uses a randomizing device about whose properties the players may hold differing beliefs. Such a device would be welcome in playing a zero-sum game: all players might believe their expected payoffs to be positive! Aumann (1987) remarks that such a result depends on "a conceptual inconsistency between the players." By permitting such inconsistencies, subjective correlated equilibrium places only weak restrictions on solutions of many games. A risk neutral equilibrium adds the nontrivial restriction that the players' risk neutral prior probabilities should be mutually consistent, as in an equilibrium of a financial market. Whenever agents are risk averse with significant prior investments in events, their true probabilities are not revealed by their preferences among financial assets, and inconsistencies among them are neither surprising nor problematic. One of the important functions of financial markets is to deal with this very situation.

As in the risk neutral case, there is more to be said about the rational solution of the game than to identify the outcomes that are jointly coherent. It is also possible to place bounds on risk neutral probabilities of events or risk neutral expectations of financial assets that depend on the outcome of the game, namely whatever bounds are determined by the system of inequalities $\boldsymbol{G}^*\boldsymbol{\pi} \geq \mathbf{0}$ that defines the convex polytope of risk neutral equilibria. These bounds are bid-ask spreads for assets that the players are jointly offering to the observer through their bets that reveal information about the rules of the game.

A simple example of the concept of risk neutral equilibrium is provided by the zero-sum game of Matching Pennies, whose payoff matrix is:

	Left	Right
Top	1, –1	–1, 1
Bottom	–1, 1	1, –1

When played by risk neutral players, the revealed-rules matrix $\boldsymbol{G}$, scaled to a maximum value of 1, is:

	TL	TR	BL	BR
1TB	1	-1	0	0
1BT	0	0	-1	1
2LR	1	0	-1	0
2RL	0	-1	0	1

This game has a unique correlated/Nash equilibrium in which the players use independent 50-50 randomization, so the graph of the set of equilibria consists of the single point (¼, ¼, ¼, ¼) in the center of the saddle.

Now suppose that both players are risk averse, and in particular assume that they have identical exponential utility functions, $U(x) = 1 - \exp(-\rho x)$, in which the risk aversion parameter is $\rho = \mathrm{LN}(\sqrt{2})$. In units of utility, the payoff matrix of the matching-pennies game is then:

	Left	Right
Top	*a*, *b*	*b*, *a*
Bottom	*b*, *a*	*a*, *b*

where $a = 1 - \sqrt{½} \approx 0.293$ and $b = 1 - \sqrt{2} \approx -0.414$. The corresponding marginal utilities of money in the vicinity of the payoffs *a* and *b* are 0.245 and 0.490, respectively, which conveniently differ by a factor of exactly 2.

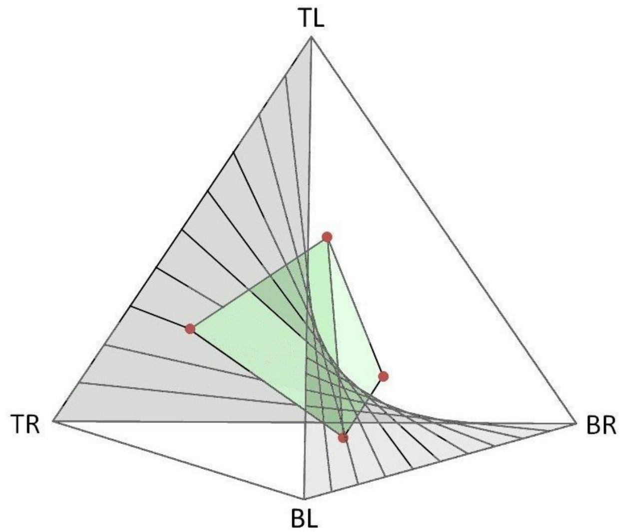
This game is constant-sum and strategically equivalent to the original one, having the same unique correlated/Nash equilibrium that uses independent 50-50 randomization. However, the rules matrix of the corresponding revealed game, $\boldsymbol{G}^*$, is *not* equivalent because of the distortions of nonlinear utility for money. It looks like this when scaled to a maximum value of 1:

	TL	TR	BL	BR
1TB	1	–1/2	0	0
1BT	0	0	–1/2	1
2LR	–1/2	0	1	0
2RL	0	1	0	–1/2

The polytope of risk neutral equilibria determined by the inequalities $\boldsymbol{G}^*\boldsymbol{\pi} \geq \mathbf{0}$ is no longer a single point. Rather, it "blows up" to a tetrahedron with these vertices:

	TL	TR	BL	BR	EV>0?
Vertex 1	2/15	4/15	1/15	8/15	1BT
Vertex 2	8/15	1/15	4/15	2/15	1TB
Vertex 3	4/15	8/15	2/15	1/15	2RL
Vertex 4	1/15	2/15	8/15	4/15	2LR

None of them lies on the saddle of distributions that are independent between {T,L} and {B,R}, so none is a Nash equilibrium of a game with these strategy sets. Each of these probability distributions satisfies 3 out of the 4 incentive constraints with equality, i.e., it assigns an expected value of zero to 3 out of the 4 rows of $\boldsymbol{G}^*$. The label of the row whose expected value is positive is shown in the rightmost column of the table. The graph of this set of neutral equilibria is shown below. The tetrahedron of risk neutral equilibria is suspended in the middle of the probability simplex, and the saddle of independent distributions cuts through its interior, a situation that would be impossible for a set of correlated equilibria of a game among risk neutral players.



The uniform distribution that is the unique equilibrium of the game when the true utility functions of the players are common knowledge lies in the interior of this polytope. When players are risk averse, the small side bets they are willing to accept do not fully reveal the between-strategy differences in utility profiles that they face in the game, so the set of risk neutral equilibria is larger than the set of correlated equilibria. This is true in general, as formalized by:

THEOREM 5: The set of correlated equilibria of a game with monetary payoffs played by risk neutral players is a subset of the set of risk neutral equilibria of the same game played by risk averse players.

Proof: If player i is risk neutral, she will accept a bet with payoff vector $(\boldsymbol{x}_{ij} - \boldsymbol{x}_{ik})\boldsymbol{e}_{ij}$, while if she is risk averse, she will accept a bet with payoff vector $((\boldsymbol{u}_{ij} - \boldsymbol{u}_{ik})/\boldsymbol{u}_{ij}')\boldsymbol{e}_{ij}$, where $u_{ij}(\boldsymbol{s}) = U_i(x_{ij}(\boldsymbol{s}))$, and $u_{ij}'(\boldsymbol{s}) = U_i'(x_{ij}(\boldsymbol{s}))$. The term $\boldsymbol{e}_{ij}$ will be ignored henceforth because it zeroes-out the same elements of both vectors. By the subgradient inequality, $U(z) < U(y) - U'(y)(y - z)$, because the value of a strictly concave function U at z must lie below the tangent to its graph at any other point y. Plugging in $y = x_{ij}(\boldsymbol{s})$ and $z = x_{ik}(\boldsymbol{s})$ yields $u_{ik}(\boldsymbol{s}) \leq u_{ij}(\boldsymbol{s}) - u_{ij}'(\boldsymbol{s})\,(x_{ij}(\boldsymbol{s}) - x_{ik}(\boldsymbol{s}))$, which rearranges to $(u_{ij}(\boldsymbol{s}) - u_{ik}(\boldsymbol{s}))/u_{ij}'(\boldsymbol{s}) \geq x_{ij}(\boldsymbol{s}) - x_{ik}(\boldsymbol{s})$, with strict inequality if $x_{ij}(\boldsymbol{s}) \neq x_{ik}(\boldsymbol{s})$. This means that $\boldsymbol{G^*} \geq \boldsymbol{G}$ pointwise, i.e., the bet that player i is willing to accept when she chooses strategy j in preference to k if she is risk neutral is weakly dominated by the bet she will accept in the same game if she is risk averse. It follows that $\boldsymbol{G}\,\boldsymbol{\pi} \geq \boldsymbol{0}$ implies $\boldsymbol{G}^*\boldsymbol{\pi} \geq \boldsymbol{0}$ for any probability distribution $\boldsymbol{\pi}$, so if $\boldsymbol{\pi}$ is a correlated equilibrium of the game played by risk neutral players, then it is a risk neutral equilibrium of the same game when it is played by risk averse players. ■

Hence, risk aversion introduces even more imprecision into the probabilistic solutions of non-cooperative games when their rules must be revealed through credible bets.

7 Rewriting the rules of the game

It was pointed out earlier, in the discussion of the battle-of-sexes game, that players could accept additional bets with an observer, beyond those that determine the rules of the game, in order to reveal more precise information about their joint beliefs. However, if they are risk neutral and have in fact implemented a Nash or correlated equilibrium, which induces a common prior distribution over outcomes of the game, they cannot both be made strictly better off through bets with each other. When players are risk averse, this is not necessarily true, and the matching-pennies game provides a good example. When played by risk averse players, it is a negative-sum game in units of utility, and for both players the unique Nash equilibrium (coin-flipping) has an expected utility that is below their status quo utility. Risk averse players would rather not play this game at all. Furthermore, player 1's marginal utility of money is greater in outcomes TR and BL (her losing outcomes) than in the other two, and vice versa for player 2. The Nash equilibrium is therefore not a *competitive* equilibrium of a financial market in which it is possible for the players to make additional bets that reveal their *solution* of the game in addition to the bets that reveal the *rules* of the game (the latter being the rows of G^*). In the context of the Nash equilibrium, it is desirable to both players to make a bet in which player 1 wins \$$x$ if TR or BL occurs and player 2 wins \$$x$ if TL or BR occurs, for any positive $x \leq 1$. Such a bet changes the rules of the game in a substantive way, but coin-flipping remains a Nash equilibrium. By choosing $x = 1$ they can even zero-out their payoffs, dissolving the game altogether. If they do not bet with each other in this fashion, but instead bet separately with an observer, there is an arbitrage opportunity for the observer that arises from the fact that, at the outset, the players' risk neutral probabilities do not agree if their true probability distributions are uniform.

8 Conclusions

The concept of coherent lower and upper previsions extends in a natural way to non-cooperative game theory, where it can be applied to the process of revealing the rules of the game as well as expressing the beliefs of the players. A rational solution of the game, from the perspective of an observer, is typically a convex set of correlated equilibria rather than a Nash equilibrium. The presence of aversion to risk changes the units of analysis from "true" subjective probabilities to "risk neutral" probabilities, as in asset pricing theory, and it typically renders the solutions even more imprecise. When risk averse players make bets with each other that reflect their beliefs about the solution of the game as well as the rules from which they started, they may be able to rewrite those rules in a mutually beneficial way, merging the concepts of strategic and competitive equilibrium

These results address some of the issues raised by Kadane and Larkey (1982) concerning the relation between game theory and subjective probability theory. The theory of game-playing presented here is a direct extension of subjective probability theory à la de Finetti, and it exploits the underappreciated common-knowledge property of de Finetti's use of bets to measure beliefs. Common knowledge of a game's rules constrains rational beliefs but in general it does not uniquely determine them, leaving room for subjective differences, particularly when players are risk averse and/or have incomplete knowledge of their own payoff functions.

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Appendix: proof of Theorem 3

The first part of (i), stating that at least one of the pure Nash equilibria must be weakly Pareto efficient, follows from the fact that each player's maximum payoff is achieved at either TL or BR for any values of c and d. To establish the other results, first observe that the pattern of signs in the game matrix of Table 10 implies that any probability distribution satisfying the incentive constraints of a correlated equilibrium will continue to satisfy them if any probability mass is shifted from either TR or BL to either TL or BR if it is feasible to do so. Whether such a probability shift yields a Pareto improvement over vertices 3, 4, or 5 depends on the values of c and d. For example, a shift of probability ε from TR to TL in the solution, if it is feasible, will increase the payoff of Row by an amount $(a-c)\ \varepsilon$ and increase the payoff of Column by ε, which is Pareto improving if $a > c$. The following table summarizes the four possible shifts of probability to TR or BL from one of the other two outcomes and the conditions under which they are feasible and Pareto improving:

	Shift 1	Shift 2	Shift 3	Shift 4
Direction of probability shift:	TR → TL	BL → BR	TR → BR	BL → TL
Feasible at vertices:	3, 4	3, 5	3, 4	3, 5
Proportional benefit to Row:	$a - c$	$1 + c$	1	a
Proportional benefit to Row:	1	b	$b - d$	$1 + d$
Pareto improving if:	$a > c$	$c > -1$	$b > d$	$d > -1$

Either shift 1 or shift 2, or both, must satisfy the Pareto improvement condition, because the positivity of a guarantees that $a > c$ must be true if $c > -1$ is not. This eliminates vertex 3 and either vertex 4 or vertex 5, or both, if they assign positive probability to an outcome from which a Pareto improving shift of probability to TR or BL can be made. Similarly, either shift 3 or shift 4, or both, must satisfy the Pareto improvement condition, which again suffices to eliminate vertex 3 and also eliminates either 4 or 5. Vertex 4 or vertex 5 therefore cannot both be Pareto efficient, and a necessary condition for vertex 4 to be Pareto efficient is that neither of its feasible probability shifts (1 and 3) should be Pareto improving, which is the case $c \geq a$ and $d \geq b$. Similarly, a necessary condition for vertex 5 to be Pareto efficient neither of its feasible probability shifts (2 and 4) should be Pareto improving, which is the case $d \leq -1$ and $c \leq -1$. These necessary (and mutually exclusive) conditions for efficiency of vertices 4 or 5 are not sufficient, though, because it is also possible for them to be dominated by a mixture of vertices 1 and 2. A mixture that assigns probability λ to TR and probability 1–λ to BR yields expected payoffs to the players of $\lambda a+(1-\lambda)(c+1)$ and $\lambda(d+1)+(1-\lambda)b$, respectively. Consider a transfer of probability mass from outcome TR (which yields payoffs of c and d, respectively) to this mixture of TL and BR. The change in Row's expected payoff is proportional to $\lambda a+(1-\lambda)(c+1) - c$, which is positive if and only if $c < a + (1-\lambda)/\lambda$. Similarly, the change in Column's expected payoff is proportional to $\lambda(d+1)+(1-\lambda)b - d$, which is positive if and only if $d < b + \lambda/(1-\lambda)$. Strong Pareto efficiency requires that there should be no λ for which the changes in both player's expected payoffs are non-negative and at least one is positive, and weak Pareto efficiency requires only that both should not be positive. Hence, strong [weak] Pareto efficiency holds if only if, for all $\lambda \in (0,1)$, both [at least one of] the conditions $c \geq a + (1-\lambda)/\lambda$ and $d \geq b + \lambda/(1-\lambda)$ are satisfied. The proof of part (iv), for vertex 5 proceeds similarly as follows. Consider a transfer of probability mass from outcome TR (which yields payoffs of 0 to both players) to the mixture of TL and BR. The change in Row's expected payoff is proportional to $\lambda a+(1-\lambda)(c+1)$, which is positive if and only if $c > -1 - \lambda/(1-\lambda)a$. Similarly, the change in Column's expected payoff is proportional to $\lambda(d+1)+(1-\lambda)b$, which is positive if and only if $d > -1 - ((1-\lambda)/\lambda)b$, and the necessary and sufficient condition for strong [weak] Pareto efficiency is that at, for all $\lambda \in (0,1)$, least one [both] of the opposite inequalities should hold. Finally, to establish the second part of (i), suppose that one of the two pure Nash equilibria is not weakly Pareto efficient, and w.l.o.g. suppose it is vertex 2. This means that both of the inequalities $a \geq c+1$ and $d+1 \geq b$ must hold. One of the requirements for efficiency of vertex 4 is that $c \geq a + (1-\lambda)/\lambda$ should hold for $\lambda= ½$, which means $c \geq a+1$, which is incompatible with $a \geq c+1$. Similarly, one of the requirements for efficiency of vertex 5 is $d \leq -1 - b$, which (in view of the fact that $b>0$) is incompatible with $d+1 \geq b$. Hence neither vertex 4 nor vertex 5 can be weakly Pareto efficient if vertex 2 is not. ■