UNTIL THE BITTER END: ON PROSPECT THEORY IN THE DYNAMIC CONTEXT

SEBASTIAN EBERT AND PHILIPP STRACK*

We show that already a small amount of probability weighting has strong implications for the application of prospect theory in the dynamic context. A naive agent will never stop a stochastic process that represents his wealth. This holds for a very large class of processes, and independently of the reference point and the curvatures of the value and weighting functions. This dynamic result is a consequence of a static result that we call *skewness preference in the small:* At any wealth level there exists an arbitrarily small gamble (which is sufficiently right-skewed) that a prospect theory agent wants to take. By choosing a proper stopping strategy the agent can always implement such a gamble and thus never stops. We illustrate the implications for dynamic decision problems such as irreversible investment, casino gambling, and the disposition effect.

KEYWORDS: Behavioral Finance, Decision Making under Risk, Disposition Effect, Irreversible Investment, Probability Weighting, Prospect Theory, Skewness Preference, Time-Inconsistency.

1. INTRODUCTION

While expected utility theory (EUT, Bernoulli (1738/1954), von Neumann-Morgenstern (1944)) is the leading normative theory of decision making under risk, cumulative prospect theory (CPT, Kahneman and Tversky (1979), Tversky and Kahneman (1992)) is the most prominent positive theory. EUT is well-studied in both static and dynamic settings, ranging from game theory over investment problems to institutional economics. In contrast, for CPT most research so far has focused on the static case. In this paper, we investigate CPT's predictions in the dynamic context and point out fundamental properties of CPT. We illustrate immediate consequences for typical dynamic decision problems such as irreversible investment, casino gambling, or the disposition effect.

Usually, CPT is characterized by four features: First, outcomes are evaluated by a utility function relative to some reference point which separates all outcomes into gains and losses.

Ebert: Institute for Financial Economics and Statistics, and Bonn Graduate School of Economics, University of Bonn, Adenauerallee 24 - 42, D-53113 Bonn, Germany; Email: sebastianebert@uni-bonn.de. Strack: Hausdorff Center for Mathematics and Bonn Graduate School of Economics, University of Bonn, Lennéstr. 43, D-53113 Bonn, Germany; E-mail: philipp.strack@gmail.com.

*This version: 15 February, 2012. Acknowledgments: We thank Nicholas Barberis, Paul Heidhues, Daniel Krähmer, Matthew Rabin, Sven Rady, and Georg Weizsäcker for helpful comments and discussions on the topic. We further thank participants of the Bonn Finance and Insurance Seminar and the 2012 DIW Behavioral Economics Workshop for helpful comments and suggestions. This project was initiated while Ebert participated in the 2nd Yale Summer School on Behavioral Finance and when Strack was a visiting researcher at Yale. Financial support from the Hausdorff Center for Mathematics (HCM) and the German Research Foundation (DGF) is gratefully acknowledged.

Second, utility is S-shaped, i.e., convex for losses and concave for gains. Third, probabilities are distorted by inverse-S-shaped probability weighting functions (one for gains and one for losses). Therefore, probabilities close to zero or one are overweighted while moderate probabilities are underweighted. Fourth, losses loom larger than gains, which is referred to as loss aversion.

A small amount of probability weighting is the single driving source this paper's results. "Small" is relative to the amount of loss aversion and will be precisely defined. Notably, the curvatures of the value and weighting functions are immaterial to our results, and so is the choice of the reference point. If some likelihood insensitivity is considered to be a fundamental element of prospect theory, then so are the results in this paper.

Our dynamic results can be traced back to a seemingly innocuous result that we call *skewness preference in the small*. At any wealth level, a CPT agent wants to take a sufficiently right-skewed binary risk which is arbitrarily small, even if it has negative expectation. We call such a risk *attractive* to the CPT agent. Therefore, a CPT agent can always be lured into gambling by offering an attractive risk. We show that such a risk *may* be small. However, depending on the value function, the risk may in fact be quite large. For the original parametrization of Tversky and Kahneman (1992) and most empirical estimates we show that there exists an attractive risk of *any* size.

A theory-free definition of risk aversion (risk seeking) at wealth level x is that any zero-mean risk is unattractive (attractive) to the agent. Therefore, skewness preference in the small implies that a CPT agent is not risk averse at any wealth level, and a symmetric result says that CPT is also risk seeking nowhere. We are not aware of a formal proof of this result, not even for the original version of CPT by Tversky and Kahneman (1992). In particular, our result implies that a small amount probability weighting eventually dominates any curvature effects on risk aversion that concave and convex parts of the value function may have: CPT does never imply risk aversion over gains and risk seeking over losses.

In a seminal paper, Barberis (2012) has revealed how the probability weighting component of prospect theory induces a time inconsistency. He provides the general intuition, and illustrates the mechanics for when gambling a 50-50 bet up to five periods. Barberis explains why naive agents (who are unaware of their time inconsistency) typically plan to follow a stop-loss strategy when entering the casino, but end up playing a gain-exit strategy. Such behavior reminds of the disposition effect pointed out by Shefrin and Statman (1985): individual investors are more inclined to sell stocks that have gained in value (winners) rather than stocks that have declined in value (losers). We will show that in more general settings, i.e., for less specific stochastic processes, prospect theory does not predict such behavior for naive agents.

In Xu and Zhou (2011) the optimal strategy of sophisticated agents who have acess to a commitment device is derived in the context of optimal stopping in continuous time. Due to the availability of commitment there is no issue of time inconsistency in this context.

In contrast to this our main result is a Theorem stating that a naive CPT agent will never stop a stochastic process representing his wealth. This holds for our general version of CPT, and for a large class of stochastic processes including geometric and arithmetic Brownian motion. In particular, the process may have arbitrary (positive or negative) drift. By planning to follow a proper stopping strategy (which is stop-loss) the agent can always implement a binary gamble which is attractive because of skewness preference in the small. Intuitively, at any point in time the agent thinks "If I loose just a little bit more, I will stop. And if I gain, I will continue." But once a loss *or* a gain has occurred, a new attractive stopping strategy will come to his mind and thus he will continue to gamble.

Differently from Barberis' five-period example with 50-50 bets, our agent will also continue to gamble if he makes a gain. In Barberis (2012), the combination of symmetric gambles and finite (very short) time horizon implies that the "casino dries out of skewness." For example, in the last period there is only a 50-50 gamble available in the casino so that skewness preference in the small does not apply. The availability of sufficiently skewed gambles is the crucial input to our limit result. A continuous, infinite time horizon setup is sufficient for it. However, we show that the result is robust to finite and/ or discrete time spaces, as long as the stochastic process allows for a rich set of possible gambling strategies. In complete markets, for example, our result will hold irrespective of the time space.

In Section 2 we will define our general version of CPT. In Section 3 we present our static result that CPT implies skewness preference in the small and point out several implications. Section 4 presents the main result that a naive agent never stops a stochastic process that represents his wealth. Section 5 discusses the implications for CPT models of casino gambling, optimal time to invest, and the disposition effect. Section 6 discusses the robustness of our result towards discrete and finite time spaces. Section 7 concludes.

2. PROSPECT THEORY PREFERENCES

We consider an agent with CPT preferences over random variables $X \in \mathbb{R}$. A CPT agent evaluates the risk X as

$$CPT(X) = \int_{\mathbb{R}_+} w^+(\mathbb{P}(U(X) > y))\mathrm{d}y - \int_{\mathbb{R}_-} w^-(\mathbb{P}(U(X) < y))\mathrm{d}y \tag{1}$$

with non-decreasing weighting functions $w^-, w^+ : [0,1] \to [0,1]$ with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$ and a value function $U : \mathbb{R} \to \mathbb{R}$ that satisfy the assumptions explained in the following.

Assumption 1 (Value function) *The value function is absolutely continuous and strictly increasing. Further,* $\lambda = \sup_{x\in\mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)} < \infty$ *exists, where* $\partial_- U(x)$ *and* $\partial_+ U(x)$ *denote the left and right derivative of* U*, respectively. W.l.o.g.* $U(0) = 0$.

Under CPT, preferences are defined on changes relative to some reference point, which is denoted by $r \in \mathbb{R}$. Typical choices for r are the status quo or some other benchmark. For example, when investing in a risky asset, r could be the return of a risk-free investment. Realizations x of X with $x < r$ are referred to as losses, and realizations $x \geq r$ are called gains. Our results hold for any $r \in \mathbb{R}$. In other words, the choice of r is immaterial to our findings. In many specifications of prospect theory, the additional assumption is made that U is differentiable everywhere, except at the reference point such that $\lambda = \frac{\partial_- U(r)}{\partial_+ U(r)}$. It is further assumed that $\lambda > 1$ and that the reflection property

$$(2) \qquad U(x) = \begin{cases} u(x-r), & \text{if } x \geq r \\ -\lambda u\left(-(x-r)\right), & \text{if } x < r \end{cases}$$

holds for some function u. $\lambda > 1$ then implies that losses loom larger than gains to the CPT agent; see Köbberling and Wakker (2005) for an analysis of the loss aversion index $\frac{\partial_- U(r)}{\partial_+ U(r)}$. We allow for non-differentiable utility because it allows to model preference over assets with non-differentiable payoffs such as option contracts. The original choice for u by Tversky and Kahneman (1992) was power utility. Since several caveats have been pointed out for the power utility parametrization, exponential utility has become another popular choice (de Georgi and Hens (2006)).[1] The final important feature of CPT is that the agent distorts the decumulative probabilities associated with gains and the cumulative probabilities associated with losses by means of respective weighting functions w^+ and w^-.

Assumption 2 (Likelihood-Insensitive Weighting Functions) *The weighting functions* w^+ *and* w^- *satisfy*

1. $\limsup_{p\to 0} \frac{w^+(p)}{p} > \lambda$.

[1]Exponential utility satisfies Assumption 1. Power utility does not suffice Assumption 1, because both partial derivatives are infinite at the reference point and thus the Köbberling-Wakker index of loss aversion is not well defined. We will treat power utility separately in Subsection 3.4. It will be seen that, for this case, even stronger results may be obtained than those based solely on Assumption 1.

2. $\limsup_{p\to 0} \frac{1-w^-(1-p)}{p} > \lambda\,.$

Note that if w^+ and w^- are differentiable at 0 and 1, then the conditions in Assumption 2 simplify to ${w^+}'(0) > \lambda$ and ${w^-}'(1) > \lambda$. If these derivatives do not exist because they approach infinity, the limit superior is infinite which is also consistent with Assumption 2. We do not require the weighting functions to be inverse-S-shaped, i.e., to start out concave and to turn convex at some point. Our assumption is weaker and much more in the spirit of Wakker and Tversky (1995), who define a connected likelihood insensitivity region bounded away from both 0 and 1. Actually, here we only require overweighting of small probabilities when associated with high outcomes; see Wakker (2010, pp. 222-233) for a comprehensive discussion of likelihood insensitivity and inverse-S-shape.

Observation 1 *Assumption 2 is satisfied by the commonly used weighting functions of Kahneman and Tversky (1979), Goldstein and Einhorn (1987), Prelec (1998), and the neo-additive weighting function. Whether the weighting function of Rieger and Wang (2006) suffices Assumption 2 depends on parameter estimates.*

Proof. See appendix. □

Rieger and Wang (2006) farsightedly proposed a weighting function with finite derivatives at 0 and 1, and showed that this ensures non-occurrence of the St. Petersburg paradox under CPT. However, it remains to be investigated whether other desirable predictions of CPT will be maintained for such weighting functions. In any case, while existing, finite derivatives at 0 and 1 are a cure to the St. Petersburg paradox, such a version of CPT may still succumb to our results.[2]

Moreover, many of our results allow to relax Assumption 2 so that they apply to any weighting function with derivatives at 0 and 1 strictly larger than one, which corresponds to a minimal departure from EUT's linear processing of probabilities.

If not noted otherwise, in the following our sole assumptions on the CPT preference functional (1) are Assumptions 1 and Assumption 2. Our point is that a small amount of probability weighting *alone* is sufficient for a fundamental property of CPT in the static case, which in turn has drastic implications for CPT in a dynamic setting.

Many of our results use binary lotteries $L \equiv L(p, b, a)$ that yield outcome b with probability $p \in (0, 1)$, and $a < b$ otherwise. Thus it is convenient to note that the CPT preference

[2]Finite derivatives may be consistent with our Assumption 2. We only require that the derivatives at 0 and 1 are larger than λ. Unfortunately, we are not aware of any parameter estimates of the Rieger and Wang (2006) weighting function.

functional (1) evaluates binary risks as

$$CPT(L) = \begin{cases} w^+(p)U(b) + (1 - w^+(p))U(a), & \text{if } r \le a \\ w^-(1-p)U(a) + w^+(p)U(b), & \text{if } a < r \le b \\ (1 - w^-(1-p))U(b) + w^-(1-p)U(a) & \text{if } b < r. \end{cases} \tag{3}$$

3. STATIC RESULTS

3.1. *Prospect Theory's Skewness Preference in the Small*

This paper starts out with a seemingly innocuous result on prospect theory preferences and small, skewed risks. We say that a risk is *attractive* or that an agent *wants to take a risk* if the CPT utility of current wealth plus the risk is *strictly* higher than the CPT utility of current wealth.

THEOREM 1 (Prospect Theory's Skewness Preference in the Small) *For every wealth level x and every $\epsilon > 0$ there exists an attractive zero-mean binary lottery $L \equiv L(p, b, a)$ with $a, b \in (-\epsilon, +\epsilon)$, i.e., L may be arbitrarily small.*

Proof. We split the proof into three cases $x > r$, $x < r$, and $x = r$. We prove the equivalent result that for all $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists a binary lottery $L \equiv L(p, b, a)$ with mean x and $a, b \in (x - \epsilon, x + \epsilon)$ such that $CPT(L) > CPT(x)$. L having mean x yields

$$x = (1-p)a + pb \;\Leftrightarrow\; p = \frac{x-a}{b-a}.$$

Proof of case 1 ($x > r$). Choose $a > r$ such that both a and b are gains. Then lottery L gives the agent a utility of $CPT(L) = w^+(p)U(b) + (1 - w^+(p))U(a)$. Therefore, the agent prefers L over x if there exist $a < x$ and $b > x$ such that

$$\begin{aligned} 0 &< \left(1 - w^+\left(\frac{x-a}{b-a}\right)\right)U(a) + w^+\left(\frac{x-a}{b-a}\right)U(b) - U(x) \\ &= (U(b) - U(a))\left(w^+\left(\frac{x-a}{b-a}\right) - \frac{U(x)-U(a)}{U(b)-U(a)}\right) \\ &= \underbrace{p\,(U(b) - U(a))}_{\ge 0}\left(\frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\right). \end{aligned} \tag{4}$$

Consider sequences $(a_n, b_n)_{n\in\mathbb{N}}$, with $a_n = x - \frac{p}{n}$ and $b_n = x + \frac{1-p}{n}$. Note that by construction

$$\begin{aligned}\frac{U(b_n) - U(a_n)}{b_n - a_n} &= \frac{U(b_n) - U(x)}{b_n - x}\frac{b_n - x}{b_n - a_n} + \frac{U(x) - U(a_n)}{x - a_n}\frac{x - a_n}{b_n - a_n} \\ &= \frac{U(b_n) - U(x)}{b_n - x}(1-p) + \frac{U(x) - U(a_n)}{x - a_n}p.\end{aligned}$$

Therefore, according to equation (4), the agent prefers lottery L over x if

$$(5) \qquad 0 < \frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a_n)}{x-a_n}}{\frac{U(b_n)-U(x)}{b_n-x}(1-p) + \frac{U(x)-U(a_n)}{x-a_n}p} =: \xi_n(p).$$

First, suppose that $\frac{w^+(p)-w^+(0)}{p-0} = \frac{w^+(p)}{p} \to \infty$ for $p \to 0$. Because the subtracted part in equation (5) is bounded for every n, equation(5) is fulfilled for sufficiently small p. Moreover, since $(a_n) \nearrow x$ and $(b_n) \searrow x$ we have $a_n, b_n \in (x-\epsilon, x+\epsilon)$ for n sufficiently large. Second, suppose $\lim_{p\to 0} \frac{w(p)}{p} = w'(0) < \infty$ exists. Since $(a_n) \nearrow x$ and $(b_n) \searrow x$, for all $p \in (0,1)$,

$$(6) \qquad \lim_{n\to\infty} \xi_n(p) = \frac{w^+(p)}{p} - \frac{\partial_- U(x)}{\partial_+ U(x)(1-p) + \partial_- U(x)p} =: \xi(p) \text{ exists.}$$

By Assumption 2,

$$(7) \qquad 0 < w^{+'}(0) - \frac{\partial_- U(x)}{\partial_+ U(x)} = \lim_{p\to 0} \xi(p).$$

Since $\xi(p)$ is continuous (Assumption 1) there exists $\tilde{p} \in (0,1)$ such that also $\xi(\tilde{p}) > 0$, i.e., $\lim_{n\to\infty} \xi_n(\tilde{p}) > 0$. Therefore, equation (5), and also $a_n, b_n \in (x-\epsilon, x+\epsilon)$, is fulfilled for $n = n(\tilde{p}, \epsilon)$ sufficiently large, i.e., $L(\tilde{p}, b_{n(\tilde{p},\epsilon)}(\tilde{p}), a_{n(\tilde{p},\epsilon)}(\tilde{p}))$ is preferred over x for sure. The proofs for $x < r$ and $x = r$ are given in the appendix. □

Corollary 1 (Unfair Attractive Gambles) *For every wealth level $x \in \mathbb{R}$ there exists an attractive, arbitrarily small binary lottery with negative mean.*

Proof. The claim follows from continuity of the CPT preference functional (Assumption 1).
□

It is straightforward to formulate a local version of Theorem 1.

Corollary 2 (Local Result) *At some given wealth level x there exists an attractive, arbitrarily small zero-mean binary lottery even if Assumption 2 is relaxed by replacing $\lambda :=$*

$\sup_{x\in\mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)}$ *with* $\frac{\partial_- U(x)}{\partial_+ U(x)}$. *If* U *is differentiable at* x*, then Assumption 2 may be further relaxed by replacing* λ *with* 1.

Proof. The claim is evident from the proof of Theorem 1, and since $\frac{\partial_- U(x)}{\partial_+ U(x)} = 1$ if U is differentiable at x. □

The intuition of the proof of Theorem 1 is that CPT implies skewness preference. Ebert (2011) illustrates that, for binary lotteries, skewness—according to both the tails and moments definitions—is exhaustively captured in the probability parameter. Therefore, we can interpret the proof of Theorem 1 as the construction of a sufficiently right-skewed fair lottery. By letting p go to zero, the binary lottery becomes more and more right-skewed. At some point the lottery is so much skewed that a CPT agent wants to take it.

Skewness preference has been of major interest in the recent economics and finance literature. Numerous empirical and experimental papers find support for skewness preference (e.g., Kraus and Litzenberger (1976) and Boyer et al. (2010) for asset returns, Golec and Tamarkin (1998) for horse-race bets, and Ebert and Wiesen (2011) in a laboratory experiment). Moreover, various economic behaviors and financial phenomena can be explained by skewness preference, e.g., casino gambling (Barberis (2012)), underdiversification in stock portfolios (Barberis and Huang (2008)), or positive expected first-day returns accompanied by negative medium-run expected returns for initial public offerings (Green and Hwang (forthcoming)). The famous coexistence of lottery and insurance demand under CPT stems from skewness preference. In many of these situations, prospect theory may do such a good job in explaining behavior because, through its probability weighting component, it implies skewness preference. Other papers have argued like this. To best of our knowledge, however, Theorem 1 is the first rigorous result that relates CPT to skewness preference.

3.2. *Prospect Theory Agents are Risk-Averse and Risk-Seeking Nowhere*

A decision-theoretic implication of Theorem 1, which is of independent interest, is on how risk aversion manifests in CPT. A theory-free definition of risk aversion (risk-seeking) at wealth level x is that any zero-mean risk is unattractive (attractive) to the agent. Numerous qualitative statements on how the curvature of the value function affects risk aversion in CPT can be found in the literature, but formal results are hard to find. A notable exception is the paper of Schmidt and Zank (2008) who characterize the curvatures of value and weighting function under which CPT exhibits strong risk aversion globally. Kahneman and Tversky (1979, p. 285) themselves noted that "[Our previous analysis] restricts risk seeking in the domain of gains and risk aversion in the domain of losses to small probabilities [...]" Here is a stronger result derived from weaker assumptions, i.e., from some probability weighting only.

Corollary 3 *At any wealth level a CPT agent is not risk-averse.*

Proof. The statement is a direct consequence of Corollary 2: At any wealth level, there exists an attractive risk. □

As in the Corollary 2, Assumptions 1 and 2 may be relaxed to obtain a tighter result locally. It is straightforward to formulate analogous versions of Theorem 1 and its corollaries on the unattractiveness of left-skewed gambles and risk-seeking under CPT. To this means, we have to assume that probabilities associated with bad outcomes are overweighted. These assumptions[3] are complementary to our Assumptions 1 and 2 and likewise fulfilled by the specifications in Observation 1. Then we have that, everywhere, there exists an arbitrarily small, left-skewed binary risk which is unattractive, and that a CPT agent is everywhere not risk-seeking. We find it striking that just some probability weighting "dominates" the impact of the curvature of the value function. In particular, our result illustrates that the intuition that the S-shaped value function of prospect theory implies risk aversion for gains and risk-seeking for losses is misleading.

3.3. *Large Risks*

Next, note that we may construct an attractive risk which is arbitrarily small. However, it must not be misunderstood that the attractive risk has to be small. Striking results on large attractive gambles have been presented by Rieger and Wang (2006) who investigate the occurrence of the St. Petersburg Paradox under CPT, and by Azevedo and Gottlieb (forthcoming) who show that risk-neutral firms can extract unbounded profits from CPT consumers. These authors construct attractive gambles that involve arbitrarily large payoffs, and thus it is intuitive that their results also require assumptions on the value function. Azevedo and Gottlieb (forthcoming) point out that for the power value function and for any attractive binary gamble L the multiple cL ($c > 1$) is also attractive. In combination with our result this then implies that there exist attractive gambles of any size. This we will make precise in the next section.

[3]Specifically, $\tilde{\lambda} = \inf_{x\in\mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)} < \infty$ must exist and 1. $\limsup_{p\to 0} \frac{w^-(p)}{p} > \lambda$ and 2. $\limsup_{p\to 0} \frac{1-w^+(1-p)}{p} > \lambda$. Evidently, these properties are also necessary for the famous inverse-S-shape, and consistent with the likelihood-insensitivity definition of Wakker and Tversky (1995). Under these assumptions one can construct an unattractive, left-skewed binary risk. The proof is similar to that of of Theorem 1, with the main difference that one must let $p \to 1$ rather than $p \to 0$ to generate left-skew.

3.4. *The Case of a S-Shaped Power Value Function*

In this section we consider a power value function which suffices the reflection property, equation (2).

ASSUMPTION 3 (S-Shaped Power Value Function) *The value function is given by*

$$U(x) = \begin{cases} (x-r)^\alpha, & \text{if } x \geq r \\ -\hat{\lambda}\,(-(x-r))^\alpha\,, & \text{if } x < r \end{cases} \tag{8}$$

with $\alpha \in (0,1)$ *and* $\hat{\lambda} > 1$.

For this very choice, the Köbberling-Wakker index of loss aversion $\frac{\partial_- U(r)}{\partial_+ U(r)}$ is not well-defined (in particular, it is not equal to $\hat{\lambda}$) because the power function has infinite derivative at 0. Therefore, Assumption 1 is not fulfilled, and thus Theorem 1 does not apply. However, we can state a similar result under a slightly different assumption on the weighting functions.

ASSUMPTION 4 *The weighting functions* w^+ *and* w^- *satisfy*

1. $\limsup_{p\to 0} \frac{w^+(p)}{p^\alpha} > \hat{\lambda}$.
2. $\limsup_{p\to 0} \frac{1-w^-(1-p)}{p} > 1$.

Note that condition 1 in Assumption 4 is stronger than condition 1 of Assumption 2. However, it is weaker than the assumption in Azevedo and Gottlieb's Proposition 1 when applied to power utility, which requires that the limit is infinite. This is the case for the weighting functions of Tversky and Kahneman (1992) and Goldstein and Einhorn (1987) under parameter restrictions that are typically fulfilled according to most empirical studies; see Azevedo and Gottlieb (forthcoming) for an elaboration. For the weighting function of Rieger and Wang (2006) our condition is not fulfilled.[4] For the weighting function of Prelec (1998) Azevedo and Gottlieb's assumption is always true.

THEOREM 2 (Skewness Preference in the Small for the S-Shaped Power Value Function) *Assume Assumptions 3 and 4 instead of Assumptions 1 and 2. For every wealth level* x *and every* $\epsilon > 0$ *there exists an attractive, zero-mean binary lottery* $L \equiv L(p,b,a)$ *with* $a, b \in (-\epsilon, \epsilon)$, *i.e.,* L *may be arbitrarily small.*

[4] Note that there is a typo in Azevedo's and Gottlieb's paper, which says that their (stronger) condition (1) is fulfilled. Actually, it is their condition (2) which is met unless the power utility parameter is equal to one. In the latter case, none of their conditions is fulfilled, but our Assumption 1 is.

Proof. Since U is differentiable everywhere except at r, the result for $x \neq r$ follows from Corollary 2. The case $x = r$ is proven in the appendix. $\square$

Power utility is differentiable everywhere except at the reference point. Therefore, note that Corollary 2, which assumes just minimal probability weighting, also applies to power utility whenever we are not at the reference point. Therefore, we need Assumption 4 exclusively to cover gambling at the reference point. Finally, let us combine Theorem 2 with the result of Azevedo and Gottlieb (forthcoming).

Corollary 4 (Skewness Preference in the Small and in the Large for the S-Shaped Power Value Function) *Assume Assumptions 3 and 4 instead of Assumptions 1 and 2. Then there exists an attractive, zero-mean binary lottery of arbitrary size.*

Proof. According to Theorem 2 there exists an attractive, arbitrarily small binary risk. According to Azevedo and Gottlieb (forthcoming) it can be scaled up to any size. $\square$

4. ON PROSPECT THEORY IN THE DYNAMIC CONTEXT

In this section we investigate the consequences of skewness preference in the small in the dynamic context. Assume that Assumptions 1 and 2 are fulfilled. Alternatively, assume the power utility case, i.e., Assumptions 3 and 4. We now define a stochastic process $(X_t)_{t\in\mathbb{R}_+}$ that could reflect the cumulated returns of an investment project, or the price development of an asset traded in the stock market. It could likewise model an agent's wealth when gambling in a casino. Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion and $(X_t)_{t\in\mathbb{R}_+}$ a Markov diffusion that satisfies

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

where we assume $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ such that there exists a unique solution with continuous paths.[5] Note that the most frequently considered processes, arithmetic and geometric Brownian motion, are covered by this definition. We denote by $\mathcal{S}$ the set of all stopping times such that the agent bases his stopping decision only on his past observations. Formally, all $\tau \in \mathcal{S}$ are adapted to the natural filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of the process $(X_t)_{t\in\mathbb{R}_+}$. At every point in time the naive prospect theory agent faces the problem of finding a stopping time $\tau \in \mathcal{S}$ that maximizes his prospect value $CPT(X_\tau, \mathcal{F}_t)$ given his information $\mathcal{F}_t$ at time

[5] $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to (0, \infty)$ are locally Lipschitz continuous Borel functions with linear growth, i.e., there exists a $K > 0$ such that $|\mu(x)|^2 + |\sigma(x)|^2 \leq K(1 + |x|^2)$.

t where

$$CPT(X_\tau, \mathcal{F}_t) = \int_{\mathbb{R}_+} w^+(\mathbb{P}(u(X_\tau - r) > t \,|\, \mathcal{F}_t))\mathrm{d}t - \int_{\mathbb{R}_-} w^-(\mathbb{P}(u(X_\tau - r) < t \,|\, \mathcal{F}_t))\mathrm{d}t\,.$$

The agent stops at time t if and only if his prospect value $CPT(X_\tau, \mathcal{F}_t)$ of any stopping time $\tau \in \mathcal{S}$ is less than or equal to what he gets if he stops immediately, which would be $CPT(X_t)$.
The probability weighting of prospect theory induces a time inconsistency. This has been pointed out by Barberis (2012), who very well illustrates the mechanics along the lines of a casino gambling example. While the agent plans to follow a certain strategy τ at the beginning, she might decide for another one once her wealth has changed. A naive agent does not anticipate that later she might deviate from her initial plan. Therefore, at every point in time, the agent looks for a strategy τ that brings her higher CPT utility than stopping immediately. If such a strategy exists, she continues to gamble—irrespective of her initial plan. In the following, we always consider such a naive agent.
Because no analytical solution is available, Barberis (2012) investigates planned and actual behavior by computing the CPT values of all possible gambling strategies that can be generated by a finite 50-50 binomial tree, for more than 8000 parameter combinations of the CPT parametrization of Tversky and Kahneman (1992). The exercise could then be repeated for other CPT parametrizations and for other stochastic processes. We now first present a general solution for the case of an infinite time horizon which is independent of the CPT specification, of the curvatures of the value and weighting functions, and of the reference point (which my change over time). It holds for the general class of stochastic processes specified above, in particular for processes with zero, positive, and negative drift. In Section 6 we discuss discrete and finite time.

Theorem 3 *The naive agent never stops.*

The intuition of the proof (given below) is to construct a stop-loss strategy, i.e., one where the agent plans to stop if the process falls a little bit (arrives at a) and plans to continue until it has risen significantly (up to b). This results in a right-skewed binary risk which the agent prefers to stopping immediately due to Theorem 1 or, in the case of a power value function, due to Theorem 2.
Proof of Theorem 3. Suppose the agent arrives at wealth x at time t, i.e., $X_t = x$. The

agent can stop and get a utility of $CPT(x)$, or she may continue to gamble. She continues to gamble if there exists a gambling strategy $\tau \in \mathcal{S}$, i.e., a stopping time such that $CPT(x) < CPT(X_\tau)$. We consider strategies $\tau_{a,b}$ with two absorbing endpoints $a < x < b$ which stop if the process $(X_t)_{t\in\mathbb{R}_+}$ leaves the interval (a, b), i.e.,

$$\tau_{a,b} = \inf\{s \geq t : X_s \notin (a, b)\} .$$

Denote with $p = \mathbb{P}(X_{\tau_{a,b}} = b)$ the probability that with strategy $\tau_{a,b}$ the agent will stop at b. Note that strategy $\tau_{a,b}$ results in a binary lottery for the agent. We first prove that the agent never stops if X_t is a martingale. For every stopping time $\tau_{a,b}$ consider the sequence of bounded stopping times $\min\{\tau_{a,b}, n\}$ for $n \in \mathbb{N}$. By Doobs optional stopping theorem (Revuz and Yor, p. 70), $\mathbb{E}(X_{\min\{\tau_{a,b},n\}}) = X_t = x$. By the theorem of dominated convergence it follows that

$$\mathbb{E}\left(X_{\tau_{a,b}}\right) = \mathbb{E}\left(\lim_{n\to\infty} X_{\min\{\tau_{a,b},n\}}\right) = \lim_{n\to\infty} \mathbb{E}\left(X_{\min\{\tau_{a,b},n\}}\right) = x .$$

Hence, $X_{\tau_{a,b}}$ implements the binary lottery $L(p, a, b)$ with expectation x. From Theorem 1 (Theorem 2) it follows that there exist $a, b \in \mathcal{I}$ such that the agent prefers the binary lottery induced by the strategy $\tau_{a,b}$ over the certain outcome x.
In the last step we prove that the naive agent never stops even if $(X_t)_{t\in\mathbb{R}_+}$ is not a martingale. Define the strictly increasing scale function $S : \mathbb{R} \to \mathbb{R}$ by

$$S(x) = \int_0^x \exp\left(-\int_0^y \frac{2\mu(z)}{\sigma^2(z)}\mathrm{d}z\right)\mathrm{d}y .$$

Define a new process $\hat{X}_t = S(X_t)$ and a new value function $\hat{U}(x) = (U \circ S^{-1})(x)$. Note that the loss aversion index of the value function $\hat{U}$ equals the loss aversion index of U because

$$\frac{\partial_-\hat{U}(x)}{\partial_+\hat{U}(x)} = \frac{\partial_-\hat{U}(x)S'(x)}{\partial_+\hat{U}(x)S'(x)} = \frac{\partial_-U(x)}{\partial_+U(x)} .$$

A CPT agent with the value function $\hat{U}$ facing the process $(\hat{X}_t)_{t\in\mathbb{R}_+}$ evaluates all stopping times exactly as a CPT agent with value function U who faces $(X_t)_{t\in\mathbb{R}_+}$. The process $\hat{X}_t = S(X_t)$ satisfies (Revuz and Yor (1999, p. 303 ff))

$$\mathbb{P}\left(\hat{X}_{\tau_{a,b}} = S(b)\right) = \mathbb{P}\left(X_{\tau_{a,b}} = b\right) = \frac{S(x) - S(a)}{S(b) - S(a)} ,$$

and hence it follows from the argument for martingales that the agent never stops. $\square$

5. APPLICATIONS

5.1. *Casino Gambling*

Our first example is the continuous, infinite time horizon analogue to the discrete, finite time setting of Barberis (2012). Let $(X_t)_{t\in\mathbb{R}_+}$ be a Brownian motion with negative drift $\mu(x) = \mu < 0$ and constant variance $\sigma(x) = \sigma > 0$, i.e.,

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \,.$$

Due to the negative drift the agent loses money in expectation if he does not stop. Further assume that the process absorbs at zero since then the agent goes bankrupt. From Theorem 3 it follows that the naive agent gambles until the bitter end, i.e., he will not stop gambling unless he is forced to due to bankruptcy. From standard results in probability theory we know that this will happen almost surely, i.e., $\mathbb{P}(X_\tau = 0) = 1$. We will compare this result to that of Barberis (2012) in Section 6.

5.2. *Exercising an American Option*

Let $(X_t)_{t\in\mathbb{R}_+}$ be a geometric Brownian motion with drift $\mu \in \mathbb{R}$ and variance $\sigma > 0$, i.e.,

$$\mathrm{d}X_t = X_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$$

The agent holds an American option that pays

$$\pi(X_t) = \max\{e^{-\alpha t}(X_t - K), 0\}$$

if exercised at time t where $\alpha > 0$ denotes the risk-free rate. Here $K \in \mathbb{R}_+$ represents the costs of investment. The American option could be interpreted as an investment opportunity, i.e., a real option (compare Dixit and Pindyck (1994)). We assume $\mu < \alpha$ to ensure that the value of the expected value maximizer is finite. The payoff $\pi(X_t)$ is incorporated into the model by simply replacing the agent's value function $U(\cdot)$ by $\hat{U}(X_t) = U(\pi(X_t))$. Here we benefit from not having assumed differentiability of the value function. The agent is allowed to exercise his option at every point in time $t \geq 0$. From Theorem 3 it follows that the agent will never exercise his option, i.e., $\tau = \infty$. As $\lim_{t\to\infty} e^{-\alpha t}(X_t - K) \xrightarrow{P} 0$ the naive prospect theory agent gets a payoff of zero even though he could get a strictly positive payoff by exercising the option immediately whenever $X_0 > K$.

5.3. *Prospect Theory Fails to Explain The Disposition Effect*

The disposition effect (Shefrin and Statman (1985)) refers to individual investors being more inclined to sell stocks that have gained in value (winners) rather than stocks that have declined in value (losers). Numerous papers have addressed this phenomenon, and some of the most immediate explanations such as transaction costs, tax concerns, or portfolio rebalancing have been formidably ruled out by Odean (1998).

Several papers have investigated whether prospect theory can explain the disposition effect. However, all of them seem to have done so without the consideration of probability weighting (Barberis (2012, footnote 26)). Formal models (without probability weighting) have been put forward just recently by Kyle et al. (2006), Barberis and Xiong (2009), and Henderson (forthcoming). The results are mixed. Some find that prospect theory can predict the disposition effect, and others that it cannot, at least not under all relevant circumstances. Barberis (2012) notes that the binomial tree in his paper, which models a casino, may likewise represent the evolution of a stock price over time. Then, naive investors may exhibit a disposition effect, even though they plan to do the opposite of the disposition effect. Our result can be related to the disposition effect in the same spirit.

We have shown that, in general, under probability weighting a naive CPT agent will sell neither losers nor winners at any time. As a consequence, a continuous time model of prospect theory with probability distortion does not predict a disposition effect for naive investors. This is especially striking as Henderson (forthcoming) shows that, in an analogous model without probability distortion, prospect theory can explain the disposition effect.

Note that the continuous time price processes such as geometric Brownian motion that are covered by our setup fit particularly well for financial market models. In any case, in the next section we show that our result also applies to a wide range of continuous or discrete, finite or infinite time horizon processes.

6. ROBUSTNESS TO DISCRETE AND FINITE TIME SPECIFICATIONS

While it may seem that our results are related to the continuous time setup, they are not. Continuous time ensures that at every point in time the strategy set of the agent is sufficiently rich. To illustrate this point consider a binomial random walk $(X_t)_{t\in\mathbb{N}}$ with jump size one and equal probability for up- and down movements. At every point in time t the agent can choose the stakesize $s_t \in [0,1]$ (as a fraction of his wealth y_t) to bet. The evolution of his wealth is then given by

$$y_{t+1} = y_t + s_t y_t (X_{t+1} - X_t)$$

with initial wealth $y_0 > 0$. The following strategy (of choosing s_t) results in any given fair binary lottery $L(p, b, a)$. Choose s_t maximal such that $y_{t+1} \in [a, b]$, i.e.,

$$\begin{aligned} s_t &= \max\{\tilde{s} \in [0,1] : (1+\tilde{s})y_t \geq a \text{ and } (1+\tilde{s})y_t \leq b\} \\ &= \min\{1 - \frac{a}{y_t}, \frac{b}{y_t} - 1\}\,. \end{aligned}$$

Due to the martingale property it follows from Doob's optional sampling theorem that the probabilities of hitting b and a that are induced by this strategy are fair, i.e., are p and $1-p$, respectively. If $L(p, b, a)$ is attractive according to either Theorem 1 or 2, then the agent will gamble with this strategy in mind. Since an attractive lottery exists at any wealth level (i.e., at any time t) the agent never stops.

The crucial point of this example is that the *time space* may be discrete if we ensure that the *strategy space* is sufficiently rich. Specifically, a global result like Theorem 1 requires that, at any time t, for any state X_t, there is at least one stopping strategy available that results in an attractive gamble. This explains why the gambling behavior documented in Barberis (2012) is different. Barberis considers behavior when gambling a 50-50 bet up to five periods. This *combination* of symmetric gambles and finite (very short) time horizon ensures that the "casino dries out of skewness." That is, at some exogenous point in time, the casino does not allow for gambling strategies any more that result in attractive gambles. The set of possible gambling strategies becomes smaller, or *coarser*, over time.

With this in mind it is immediate that we can also have never stopping with a finite time space. To this means, the casino must be able to offer a sufficiently skewed gamble (which is attractive according to either Theorem 1 or 2) in a single period, i.e., in the final period. This is just a less subtle way (compared to allowing for an infinite time horizon) to enrich the strategy space. To illustrate this point we give a numerical example in Subsection A.4 in the appendix. There we assume the original finite, discrete time setting introduced by Barberis (2012), and simply change the probability of an up-movement in the binomial tree from 1/2 to 1/37. We show that an agent with CPT preferences of Tversky and Kahneman (1992) and the parameter estimates from that paper never stops gambling for any finite or infinite horizon.

If a casino cannot offer one-shot gambles with sufficient skewness, then several periods of gambling might be necessary to construct an attractive gamble. The number of periods will depend both on the maximal skewness of available one-shot gambles, and on the particular CPT specification and parameter choices. Then, we will have an *endgame effect* as in Barberis (2012). The analysis of this effect is extremely insightful to understand the interaction of

probability weighting, time-inconsistency, and naiveté.
However, our result shows that such effects will vanish if a casino can offer a rich set of gambling strategies. Likewise, endgame effects will vanish in financial markets that offer a variety of products. In complete markets, in particular, we have that a naive CPT agent never stops investing into a risky asset.
These thoughts point to a fundamental challenge for the application of prospect theory in finite time horizon models of gambling or investment. In such models it will be hard to disentangle whether the conclusions stem from the particular process assumed, from the particular prospect theory preference specification—or from other features of the model one might actually be interested in. In particular, the number of periods is important because they influence the richness of gambling strategies. In other words, a tough question is to what extent results are due to an endgame effect, which, according to the results of this paper, will vanish when enriching the strategy space.

7. CONCLUSION

We set up a very general version of cumulative prospect theory (CPT) and point out fundamental implications for that theory that stem from probability weighting alone. The reference point and the curvatures of the weighting and value functions are immaterial to our results. We first prove that probability weighting implies skewness preference in the small. At any wealth level, a CPT agent wants to take a sufficiently right-skewed binary risk that is arbitrarily small, even if it has negative expectation. To best of our knowledge, this is first rigorous result that relates CPT to skewness preference. A corollary is that CPT agents are not risk-averse, even if, for example, the value function is concave everywhere. While we prove the existence of small attractive risks, under additional assumptions on the value function we show that attractive risks may, in fact, be quite large. For the power value function of Tversky and Kahneman (1992) we show that for typical parameter estimates there exist attractive risks of arbitrary size.
These static results have consequences for CPT in the dynamic context. We investigate the predictions of probability weighting for a naive agent who is unaware of his time-inconsistency, which is induced by probability weighting. Such a naive agent will never stop a stochastic process that represents his wealth. The implications of this result are very extreme. Naive agents will gamble in a casino until the bitter end, i.e., they will go bankrupt almost surely. They will never exercise an American option, even if it is profitable to do so right from the beginning. CPT does not predict the disposition effect for naive agents. These results are formulated for a continuous, infinite time horizon.

Then we illustrate that the results extend to discrete time, as long as the space of stopping strategies that can be generated from the process is sufficiently rich. Likewise, we may also allow for finite time. If the time space is such that the set of stopping strategies is coarse in the sense that it does not allow to adopt stopping strategies that result in small, skewed gambles, then our result will not apply. However, casinos and even more financial markets allow for a very rich set of gambling and investment strategies. In complete markets, in particular, our never stopping result always applies.

In finite time there will be an endgame effect, which will be particularly pronounced for coarse, discrete processes. Then the set of available stopping strategies decreases with every time step. This may lead to interesting observations on the planned and actual behavior of naive agents as has been illustrated by Barberis (2012). Generally, however, it is hard to disentangle whether the conclusions stem from an endgame effect, i.e., are particular to the process assumed—or whether the conclusions stem from other features of the model, features one might actually be interested in. This may be a drawback for the application of CPT in finite time horizon models of naive behavior. Infinite time, on the other hand, yields very extreme predictions always. Therefore, the results of this paper are fundamental to any paper that investigates prospect theory's predictions for naive agents in dynamic decision models.

APPENDIX A

A.1. *Proof of Observation 1.*

Here we show that all commonly used weighting functions exhibit likelihood insensitivity according to our Assumption 2. Most results are not new, but we think that the following collection is convenient, and we are not aware of any citable source.

The weighting function of Kahneman and Tversky (1979) given by

$$w^+(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}$$

is differentiable on $(0, 1)$, and the derivative is given by

$$w'(p) = \delta p^{\delta-1} \left(p^\delta + (1-p)^\delta\right)^{-\frac{1}{\delta}} \left[1 + \frac{p^\delta (1-p)^{\delta-1}}{p^\delta + (1-p)^\delta}\right].$$

For $\delta \geq 0.28$ the function is strictly increasing. For $\delta > 1$ the function is S-shaped while for $\delta = 1$ we have $w(p) = p$. The interesting parameter range thus is $\delta \in (0, 1)$ for which w is increasing and likelihood insensitive. For $\delta \in (0, 1)$ it is easy to see that

$$\lim_{p \searrow 0} w'(p) = \lim_{p \nearrow 1} w'(p) = +\infty.$$

The weighting function of Prelec (1998) is given by

$$w(p) = (\exp(-(-\ln(p))^a))^b \tag{9}$$

with both a and b strictly positive. According to Wakker (2010), p. 207, "Ongoing empirical research suggests that $a = 0.65$ and $b = 1.05$ [...] are good parameter choices for gains." These choice implies a strong likelihood insensitivity. A special case of the function axiomatized by Prelec is for $b = 1$:

$$w(p) = \exp(-(-\ln(p))^a).$$

We show that w is likelihood insensitive if $a < 1$. Similar arguments imply an S-shape of w if $a > 1$. For $a = 1$ the function has power form and is thus either convex ($b > 1$), linear, ($b = 1$), or concave $b < 1$, and also not sufficing Assumption 2. The derivative of the general version, given by equation (9), is given by

$$w'(p) = \frac{ab}{p}\left(-\ln(p)\right)^{a-1} \cdot w(p).$$

It is straightforward that $\lim_{p\nearrow 1} w'(p) = +\infty$ for $a < 1$. To compute $\lim_{p\searrow 0} w'(p)$, substitute $x = -\log(p)$, and observe that

$$\begin{aligned} w'(p) &= \frac{ab}{p}(-\log(p))^{a-1}(\exp(-(-\log(p))^a))^b = \frac{ab}{\exp(-x)}x^{a-1}\exp(-b \cdot x^a) \\ &= a \cdot b \cdot x^{a-1}\exp(x - b \cdot x^a) \\ &= a \cdot b \cdot x^{a-1}\exp(x(1 - b \cdot x^{a-1})). \end{aligned}$$

Note that $p \to 0$ as $x \to \infty$. The expression $\lim_{x\to\infty} x(1-\frac{b}{x^{1-a}})$ goes to infinity if and only if $a < 1$. Moreover,

$$\frac{\exp(x(1 - b \cdot x^{a-1}))}{x^{1-a}}$$

goes to infinity as x^{1-a} is only of polynomial growth while $\exp(x(1 - b \cdot x^{a-1}))$ is of exponential growth.

The weighting function of Goldstein and Einhorn (1987) is defined by

$$w(p) = \frac{bp^a}{bp^a + (1-p)^a}$$

for $a > 0$ and $b > 0$. According to Wakker (2010), p. 208, "The choices $a = 0.69$ and $b = 0.77$ fit commonly found data well." These parameter choices imply a mild inverse-S-shape. The interesting parameter range for both a and b is $(0, 2)$. $a = 1$ and $b = .77$ imply linearity. For $b = 0.77$ fixed, $a > 1$ implies S-shape, and for $a < 1$ the function is inverse-S-shaped, with likelihood insensitivity decreasing (i.e., stronger inverse-S-shape) for $a \searrow 0$.

The derivative is

$$\begin{aligned}
w'(p) &= \frac{abp^{a-1}\left(bp^a + (1-p)^a\right) - bp^a\left(bap^{a-1} + a(1-p)^{a-1}(-1)\right)}{\left(bp^a + (1-p)^a\right)^2} \\
&= \frac{abp^{a-1}\left[bp^a + (1-p)^a - bp^a + p(1-p)^{a-1}\right]}{\left(bp^a + (1-p)^a\right)^2} \\
&= \frac{abp^{a-1}\left[(1-p)^{a-1}\left((1-p) + p\right)\right]}{\left(bp^a + (1-p)^a\right)^2} \\
&= \frac{abp^{a-1}(1-p)^{a-1}}{\left(bp^a + (1-p)^a\right)^2}.
\end{aligned}$$

The following holds for arbitrary $b > 0$. For $0 < a < 1$ we have

$$\lim_{p \searrow 0} w'(p) = \lim_{p \nearrow 1} = +\infty,$$

which indicates a likelihood insensitive weighting function. For $b > 0$ and $a = 1$,

$$\lim_{p \searrow 0} w'(p) = b \text{ and } \lim_{p \nearrow 1} = 1.$$

For $b > 0, a > 1$,

$$\lim_{p \searrow 0} w'(p) = \lim_{p \nearrow 1} = 0,$$

which is consistent with an S-shaped weighting function. The neo-additive weighting function is defined for a, b positive, $a + b \leq 1$, $w(0) = 0$ and $w(1) = 1$, and for $p \in (0, 1)$:

$$w(p) = b + ap.$$

That is, this function is (in general) discontinuous in 0 and 1 and linear on the interior of its domain. Therefore, it is likelihood insensitive according to our Assumption 2 except for $b = 0$ and $a = 1$.

Finally, let us consider the weighting function proposed by Rieger and Wang (2006) which can be calibrated such that Assumption 2 is not fulfilled. For $a, b \in (0, 1)$ it is given by

$$w(p) = \frac{3 - 3b}{a^2 - a + 1}\left(p^3 - (a+1)p^2 + ap\right) + p$$

with derivatives at 0 and 1

$$w'(0) = \frac{3 - 3b}{a^2 - a + 1}a + 1 \text{ and } w'(1) = \frac{3 - 3b}{a^2 - a + 1}(1 - a) + 1.$$

Moreover, it is easy to show that

$$\frac{\partial}{\partial a}w'(0) > 0,\ \frac{\partial}{\partial a}w'(1) < 0, \frac{\partial}{\partial b}w'(0) < 0 \text{ and } \frac{\partial}{\partial b}w'(1) < 0$$

which implies that

$$\sup_{a,b\in(0,1)} w'(0) = \lim_{b\to 0}\lim_{a\to 1} w'(0) = 4 \text{ and } \sup_{a,b\in(0,1)} w'(1) = \lim_{b\to 0}\lim_{a\to 0} w'(1) = 4.$$

It then easily follows that $w'(0)$ and $w'(1)$ may take any value in $(0, 4)$. The smaller b, the more pronounced is the inverse-S shape of w and also the steeper are the functions at $0, 1$. By construction, $w(a) = a$, and the derivative at 0 (1) is increasing (decreasing) in a. Thus a allows to account for different overweighting of good- and bad-outcome probabilities. Generally, the higher b the more likely our Assumption 2 is fulfilled. Unfortunately, we are not aware of any empirical estimates of the Rieger-Wang weighting functions. Moreover, Azevedo and Gottlieb (forthcoming) show that their unbounded profits paradox emerges for this function in combination with both power and exponential utility.

A.2. *Full Proof of Theorem 1*

Proof of case 2 ($x < r$.) Choose $b < r$ such that both a and b are losses. In that case, lottery $L = L(p, b, a)$ secures the agent a utility of

$$CPT(L) = (1 - w^-(1-p))U(b) + w^-(1-p)U(a)$$

with $1 - p = \frac{b-x}{b-a}$. Therefore, the agent continues to gamble if there exist $a < x$ and $b > x$ such that

$$\begin{aligned}
0 &< \left(1 - w^-\left(\frac{b-x}{b-a}\right)\right)U(b) + w^-\left(\frac{b-x}{b-a}\right)U(a) - U(x)\\
&= U(b) - U(a) + U(a) - U(x) - w^-\left(\frac{b-x}{b-a}\right)(U(b) - U(a))\\
&= (U(b) - U(a))\left(1 - w^-\left(\frac{b-x}{b-a}\right) + \frac{U(a) - U(x)}{U(b) - U(a)}\right)\\
&= \underbrace{(1-p)\,(U(b) - U(a))}_{\geq 0}\left(\frac{w^-(1) - w^-(1-p)}{1-p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\right) \qquad (10)
\end{aligned}$$

which is the analogue to equation (4). Therefore, similar to the proof of case 1, according to equation (10), the agent prefers lottery $L(p, b_n, a_n)$ over x if

$$0 < \frac{w^-(1) - w^-(1-p)}{p} - \frac{\frac{U(x)-U(a_n)}{x-a_n}}{\frac{U(b_n)-U(x)}{b_n-x}(1-p) + \frac{U(x)-U(a_n)}{x-a_n}p} =: \xi_n(p), \qquad (11)$$

which is the analogue to equation (5). The proof continues similar to that of case 1.

Proof of case 3 ($x = r$.) Consider $x = r$ such that a is a loss and b is a gain. In that case, lottery $L(p, b, a)$ secures the agent a utility of

$$CPT(L) = w^-(1-p)U(a) + w^+(p)U(b)\,.$$

Note that, since $x = r$ by definition $U(x) = U(r) = 0$. Therefore, the chooses L over x if there exist $a < x$ and $b > x$ such that

$$
\begin{aligned}
0 &< w^-(1-p)U(a) + w^+(p)U(b) - U(x) \\
&= w^+(p)\left(U(b) - U(a)\right) + \left(U(a) - U(x)\right)\left(w^-(1-p) + w^+(p)\right) \\
&= \left(U(b) - U(a)\right)\left(w^+(p) - \frac{U(x) - U(a)}{U(b) - U(a)}\left(w^-(1-p) + w^+(p)\right)\right) \\
&= \underbrace{p\left(U(b) - U(a)\right)}_{\geq 0}\left(\frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\left(w^-(1-p) + w^+(p)\right)\right) . \qquad (12)
\end{aligned}
$$

First suppose $\frac{w^+(p)}{p} \to \infty$ for $p \to 0$ then the condition follows from the fact that $w^-(1-p) + w^+(p) \leq 2$ two by definition. Suppose ${w^+}'(0)$ exists then

$$
\lim_{p \searrow 0} w^-(1-p) + w^+(p) \leq 1 + \lim_{p \searrow 0} w^+(p) = 1 .
$$

Consequently, the sufficient limit condition for gambling is like in case 1, equation (7).

A.3. *Full Proof of Theorem 2*

Suppose that a_n and b_n are is in the proof of Theorem 1. For the power-S-shaped value function it is easily seen that

$$
\frac{U(x) - U(a_n)}{x - a_n} = \frac{0 + \lambda\left(\frac{p}{n}\right)}{\left(\frac{p}{n}\right)^\alpha} = \lambda n^{1-\alpha} p^{\alpha - 1}
$$

and

$$
\frac{U(b_n) - U(a_n)}{b_n - a_n} = \frac{\left(\frac{1-p}{n}\right)^\alpha + \lambda\left(\frac{p}{n}\right)^\alpha}{\frac{1-p}{n} + \frac{p}{n}} = n^{1-\alpha}\left((1-p)^\alpha + \lambda p^\alpha\right) .
$$

Hence,

$$
\frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}} = \lambda \frac{p^{\alpha-1}}{\left((1-p)^\alpha + \lambda p^\alpha\right)} .
$$

Therefore, according to equation (12), L is attractive if

$$
\begin{aligned}
\Longleftrightarrow 0 &< p(U(b) - U(a))\left(\frac{w^+(p)}{p} - \lambda \frac{p^{\alpha-1}}{\left((1-p)^\alpha + \lambda p^\alpha\right)}\left(w^-(1-p) + w^+(p)\right)\right) \\
&= \left(\frac{w^+(p)}{p^\alpha} - \lambda \frac{1}{\left((1-p)^\alpha + \lambda p^\alpha\right)}\left(w^-(1-p) + w^+(p)\right)\right) .
\end{aligned}
$$

Since $\lambda > 1$, similarly to the proof of Theorem 1, case 3, it follows that

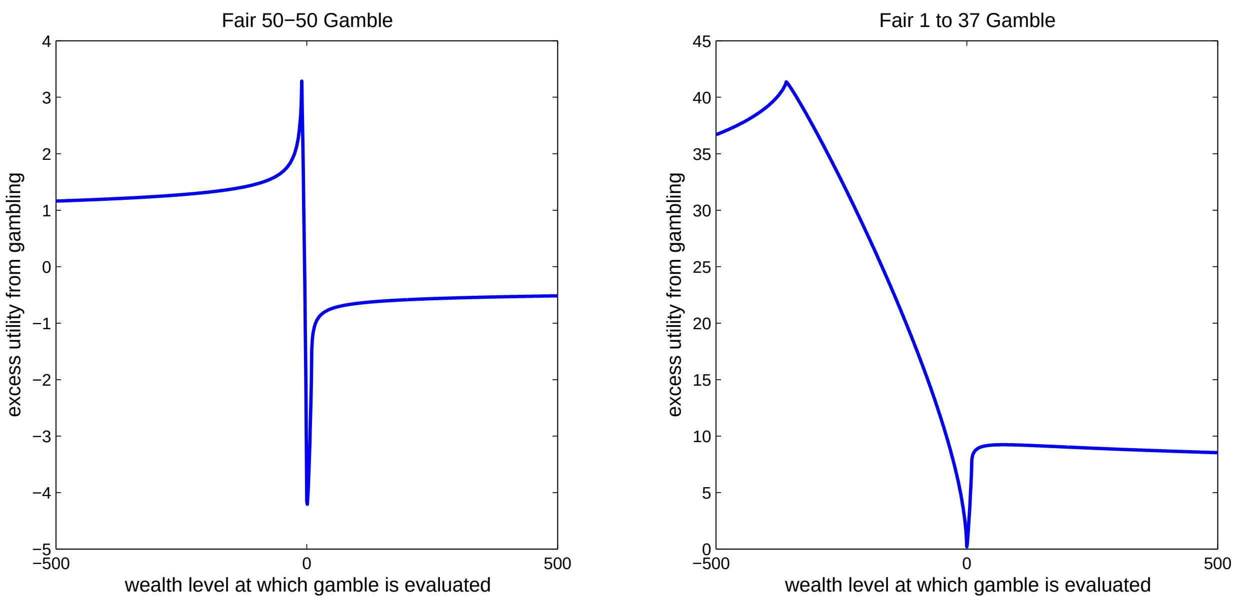
$$
\lim_{p \to 0} \frac{w^+(p)}{p^\alpha} > \lambda
$$

is a sufficient condition for gambling. □

A.4. *Example for Never Stopping in Discrete and Finite Time*

Consider the five-period binomial decision tree of Barberis (2012). Assume a casino that offers a fair version of French Roulette. We assume a fair casino to be close to the model of Barberis (2012). Then the basic gamble considered by Barberis is the fair analogue to a bet on Red or Black, which occur with equal probability. Now suppose the agent can also bet on a single number, which occurs with probability $\frac{1}{37}$. Consider an agent who only considers to bet 10 units of money on a single number. He is not even able to form a gambling strategy over several periods. This implies a rather coarse strategy space, a feature which is actually working against our never stopping result. However, the basic gamble is skewed whereas the basic gamble Barberis (2012) is symmetric. Let $(X_t)_{t\in\mathbb{R}_+}$ be the binomial random walk that represents

FIGURE 1.— Gambling Utility for a Symmetric and a Skewed gamble



This figure shows the excess utility an agent gains from gambling (over not gambling) for different wealth levels. The left panel shows the utility from gambling a fair 50-50 bet, while the right panel shows the utility from gambling a fair 1 to 37 bet. The agent is a CPT maximizer with the parametrization of Tversky and Kahneman (1992) with parameters given by $\alpha = 0.88$, $\delta = 0.65$, and $\lambda = 2.25$. The agent's reference point is 0.

his wealth. It increases by 360 with probability $\frac{1}{37}$ and decreases by 10 with probability $\frac{36}{37}$, starting at some level $X_0 \in \mathbb{R}$, i.e.,

$$\begin{aligned} \mathbb{P}(X_{t+1} = X_t + 360) &= \frac{1}{37} \text{ and} \\ \mathbb{P}(X_{t+1} = X_t - 10) &= \frac{36}{37}. \end{aligned}$$

The agent is forced to stop in the final period T, which is exogenous, or if the random walk reaches zero. Suppose the agent has CPT preferences given by the original parametrization of Tversky and Kahneman (1992) with parameters as estimated by the authors. Figure 1 plots the excess utility from gambling for the two basic gambles described above. For the 50-50 gamble (left panel), gambling is attractive over the area of losses, and unattractive at the reference point and thereafter. This fits with the common intuition of risk seeking over losses and risk aversion over gains, which is induced by the S-shaped value function.
Note that the probability weighting component has not much grip when evaluating 50-50 gambles. However, the right panel shows that gambling the skewed basic gamble is attractive *everywhere*. The lowest utility from gambling is at the reference point, but this utility is still positive (the exact value is +0.56). Therefore, at any node of the binomial tree, the agent will want to gamble. That is, the agent never stops even though we have finite time with an arbitrary number of gambling periods and a rather limited strategy space. Only one basic gamble is available, but this gamble is sufficiently skewed to be attractive to this very CPT agent. A stop-loss plan would grant even higher utility to the agent, but the one-shot gamble is attractive in itself already.

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