

## **Probabilistic Risk Attitudes and Local Risk Aversion: a Paradox\***

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**Abstract:** Prominent theories of decision under risk that challenge expected utility theory model risk attitudes at least partly with transformation of probabilities. This paper shows how attributing local risk aversion to attitudes towards probabilities can produce extreme probability distortions that imply paradoxical risk aversion.

**Keywords:** risk aversion, probability transformation, calibration, reference dependence, loss aversion

### **1. Introduction**

The first paradox to challenge expected utility theory was offered by Allais (1953). The Allais patterns violate the independence axiom, which gives the expected utility functional its idiosyncratic feature of linearity in probabilities. In order to evade the Allais paradox, theories of decision under risk that relax the independence axiom were developed (see Starmer, 2000 for an accessible presentation).

The idea of representing risk aversion with nonlinear probability transformations originated in the psychology literature about mid-twentieth century (Preston and Baratta, 1948; Edwards, 1954) and entered the economics literature in late seventies (Handa, 1977; Kahneman and Tversky, 1979; Quiggin, 1982). Some early models of probability weighting (Handa, 1977; Kahneman and Tversky, 1979) were shown to violate first order stochastic dominance. Subsequent models with rank dependence of prizes avoid that problem (Tversky and Kahneman, 1992; Quiggin, 1993). Further development and applications of rank dependent models, and other alternatives to expected utility, have continued to the present. Wakker (2010) provides a comprehensive presentation of the literature. Rank dependent utility models have been relatively successful in explaining several behavioral ‘anomalies’ that have been observed in the laboratory and in the field which accounts for their recent widespread use in the field applications.

This paper, however, is concerned with the implications of probabilistic sensitivity and rank dependence for risk aversion. I argue that attributing local (with respect to probabilities)

risk aversion to attitudes towards probabilities can produce paradoxical risk aversion. This calls into question the ability of rank dependent utility models to rationalize risk aversion at small and large probabilities.

## 2. Risk Aversion as Attitude toward Probabilities: An Example of the Paradox

In order to provide an intuition for the main result in the paper, this section presents an illustrative example of risk-avoiding choices that provide a challenge to modeling risk behavior with probabilistic attitudes.

Suppose that an individual is offered a choice between two prospects represented by urn  $S_g$  and urn  $R_g$ . Urn  $S_g$  contains 100 balls in the following composition: 10 white balls,  $g$  green balls and  $b$  black balls. Each white ball pays \$1000, each green ball pays \$5000 and each black ball pays \$0. Urn  $R_g$  is constructed from urn  $S_g$  by replacing 10 white balls with 5 green balls and 5 black balls. Table 1 summarizes all the information on prizes and compositions of balls for the two urns.

**Table 1. Illustration of composition of urns  $S_g$  and  $R_g$  ;  $g$  is from  $\{0,5,\dots, 90\}$**

Ball Prizes	\$5000	\$1000	\$0	
Ball Colors	Green	White	Black	Total
Ball composition in urn $S_g$	$g$	10	$b$	100
Ball composition in urn $R_g$	$g+5$	0	$b+5$	100

The decision problem the individual is facing is the choice between urn  $S_g$  that offers prizes \$5K, \$1K and \$0 with probabilities  $g/100$ ,  $10/100$  and  $b/100$  and a relatively riskier urn  $R_g$

that offers only extreme prizes, \$5000 and \$0 with probabilities  $(g+5)/100$  and  $(b+5)/100$ .<sup>1</sup> Figure 1 illustrates, in the probability triangle, pairs of two prospects offered by urns  $S_g$  and  $R_g$  for different combinations of green and black balls.

What might be the decision of our individual? Cox, Sadiraj, Vogt and Dasgupta (2012) reports that in their experiments the majority of subjects, who were asked to make similar choices as our individual, were either indifferent or preferred the safer prospect,  $S_g$  over the riskier prospect  $R_g$  for all compositions of green and black balls that they faced. So suppose that our individual is like the majority of subjects in the Cox et al. experiments and he weakly prefers urn  $S_g$  to urn  $R_g$  for all  $g$  from  $\{0,5,\dots,90\}$ .<sup>2</sup> Note that, in the probability triangle, such choices are expected wherever linear (but not necessarily parallel) indifference curves are steeper than the  $45^\circ$  line.

What are implications for rank dependent models if these risk avoiding choices are attributed (partly) to sensitivity towards probabilities? The implications are potentially extreme distortions of probabilities that produce paradoxical risk aversion at large probabilities as illustrated below. If  $v(\cdot)$  denotes the value function defined over the prizes<sup>3</sup> and normalized such as  $v(0) = 0$  and  $f(\cdot)$  denotes the transformation of decumulative probabilities then it can be shown (see the proof of Proposition A.1 in the Appendix) that indifference or preference for urn  $S_g$  over  $R_g$  when the number of green balls is from  $\{g, g+5, \dots, g+5k\}$  for some integer  $k(\leq 18 - g/5)$  reveals that

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<sup>1</sup> Formally, urn  $R_g$  is prospect  $\{\$5K, 0; p, 1-p\}$ ,  $p=(g+5)/100$  whereas urn  $S_g$  is prospect  $\{\$5K, \$1K, 0; p-1/2n, 1/n, 1-p-1/2n\}$ ,  $n=10$ . All prospects R and S considered in this paper are of these types.

<sup>2</sup> The supposition that the safer prospect is weakly preferred for all  $g$  from  $\{0, 5, \dots, 90\}$  is made here for simplicity of exposition; section 3 provides general results for cases when the weak preference for the safer urn is observed only for some subset of  $\{0,5,\dots,90\}$  and strict preference for the riskier urn is observed outside the subset. Note that such preferences cannot be rationalized by expected utility.

<sup>3</sup> In this paper the value function  $v(\cdot)$  is defined over prizes. For the terminal wealth model, as in rank dependent utility model,  $v(y) = u(w + y)$ ; for the income model, as in cumulative prospect theory,  $v(y) = u(y)$ .

$$(1) \quad f\left(\frac{g+5(k+2)}{100}\right) - f\left(\frac{g+5(k+1)}{100}\right) \geq q^{k+1} \left[ f\left(\frac{g+5}{100}\right) - f\left(\frac{g}{100}\right) \right]$$

where  $q = -1 + v(5K) / v(1K)$ ; 1K and 5K, respectively, denote \$1000 and \$5000. Inequality (1) shows that the slope of the transformation of decumulative probabilities increases geometrically and therefore severe underweighting of probabilities is expected when  $q > 1$ .<sup>4</sup> An implication of such preferences is for example  $f(0.5) \leq (q^{10} - 1) / (q^{20} - 1)$ . If as in Abdellaoui et al. (2007), we set  $v(5K) / v(1K) \geq 3$  then the last inequality implies  $f(0.5) \leq 0.00098$ . This extreme probability distortion and sub-additivity of  $v(\cdot)$  imply preference for a sure amount of \$20 over a prospect that offers \$20,000 or 0 with equal probability.<sup>5</sup>

## 2.b Payoff Scale Invariance

The example in this section used payoffs of \$5000, \$1000, and \$0. Inspection of statement (1) reveals that the only way in which the valuation of prizes enters the inequalities is through  $q = [v(5K) / v(1K)] - 1$ . Hence, irrespective of the size of payoffs (whether they are very large or very small or moderate in size), weak preference for urn  $S_g$  over a range of green balls implies the same paradoxical risk aversion for all payoffs with the same ratio of valuation  $q$ . For example, the prizes can be \$14 (for a green ball), \$4 (for a white ball) and \$0 (for a black ball) and risk aversion implications are similar for corresponding prizes that involve millions as long as the valuation of \$14 (or \$14 million) is more than twice the valuation of \$4 (or \$4 million). Cox et al. (2012) report on an experiment conducted in Atlanta where payoffs \$14/\$4/0 were used. Arguably, at these small stakes the utility should be approximately linear and therefore  $q$  is expected to be larger than 2. The estimated percentage of subjects who revealed weak preference for the three outcome lottery  $S_i$  over lottery  $R_i$ , ( $i=1, \dots, 9$ ) is 74% which is similar to 72% figure

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<sup>4</sup> The supposition that the value of \$5K is more than twice the value of \$1K is consistent with estimates reported in Abdellaoui, Bleichrodt and L'Haridon (2008). These papers offer appealing preference-based methods for measuring utility of (positive and negative) prizes under prospect theory; reported measurements of utility were robust to probabilities used in elicitation. In Abdellaoui et al. (2008) the reported estimated (mean) power exponent on the gain domain is 0.86. Since 5K and 1K are within the range of payoffs in Abdellaoui et al. (2008) study, our numerical illustrations will build on the value of 5K being at least 3 times the value of K, which is satisfied for the power estimates in their study.

<sup>5</sup> Verify that  $v(20000)f(0.5) \leq 1000v(20)f(0.5) < v(20)$  where the first inequality follows from subadditivity of  $v(\cdot)$  whereas the second inequality follows from  $f(0.5) < 0.00098$ .

reported (in case of higher stakes) for another experiment conducted in Calcutta with payoffs 400/80/0 in rupees.

### 3. Paradoxical Probabilistic Sensitivity and Risk Aversion

This section contains the main result of the paper. It reports on implausible implications that follow from attributing risk-aversion to attitudes towards probabilities. All proofs are collected in the appendix.

Let  $L = \{x_j; p_j\}_{j=1,\dots,n}$  denote a prospect with  $n$  prizes: it pays  $x_j$  with probability  $p_j$ . As usual, let the outcomes be rank-ordered from best to worst. The rank dependent utility of prospect,  $L$  is

$$(2) \quad U(L) = \sum_{j=1,\dots,n} v(x_j)(f(P_j) - f(P_{j-1}))$$

where:  $P_j = \Pr(x: x \geq x_j)$ ;  $f(\cdot)$  is the transformation of decumulative probability and  $v(\cdot)$  is a strictly increasing valuation function defined over prizes. Without any loss of generality the valuation of prize \$0 is normalized to 0,  $v(0) = 0$ . Over relative large gains it is assumed, consistent with the literature, that function  $v(\cdot)$  is subadditive.

I am concerned herein with individual preferences over prospects  $R_i = \{h, 0; p_i\}$ , and  $S_i = \{h, m, 0; p_i - 1/2n, 1/n\}$  where  $p_i = i/2n$ ,  $i = 1, \dots, 2n-1$  and  $n \in \mathbb{N}$ . In words, the three outcome lottery  $S_i$  pays  $h$  with probability  $(i-1)/2n$ ,  $m$  with probability  $1/n$  (that does not depend on  $i$ ) and 0 otherwise whereas the two outcome lottery  $R_i$  pays  $h$  with probability  $i/2n$  and 0 otherwise. Figure 1 illustrates such pairs of lotteries in the probability triangle. Note that all two-outcome lotteries,  $R_i$  are on the hypotenuse whereas all three-outcome lotteries,  $S_i$  lie on a line parallel with the hypotenuse at a distance  $1/n$ . Note also that lines that join lotteries  $R_i$  and  $S_i$  are parallel with slope 1. So, wherever in the probability triangle indifference curves are not flatter than the  $45^\circ$  line lotteries  $S_i$  are weakly preferred to lotteries  $R_i$ .

The theorem below states implications of a risk avoiding pattern for the transformation of probabilities and paradoxical risk aversion that follows from it. Part (a) of the theorem offers an upper bound on the difference between the transformed probabilities of any  $p \in (p_*, p^*)$  and  $p_*$  that follows from weak preference for  $S_i$  over  $R_i$  for all pairs  $i \in (2np_*, 2np^*) \cap \mathbb{N}$ ; that is for all

pairs  $i$  with  $R_i$  from a “connected” subset of the hypotenuse in the probability triangle. Part (b) states risk aversion implications that follow from part (a).

The following standard notation is used:  $\succeq$  for weak preference,  $\succ$  for strong preference,  $\lceil x \rceil$  for the smallest integer larger than  $x$ ,  $\lfloor x \rfloor$  largest integer smaller than  $x$ . In addition, hereafter whenever no confusion is expected symbols  $q$  and  $\kappa$  will be used as short notations for functions  $q = v(h)/v(m) - 1$ ,  $\kappa(q, t, s) = 1 + \frac{\sum_{j=0}^t q^{j+1}}{\sum_{i=0}^s q^{-i}}$ .

**Theorem:** Let prizes  $h > m > 0$  and  $n \in N$  be given. Suppose that  $q > 1$  and  $S_i \succeq R_i$ , for all  $p_i = i/2n \in (p_*, p^*)$ . Then

a.  $\forall p \in (p_*, p^*)$

$$(*) \quad f(p) - f(p_*) \leq \frac{1}{\kappa} [f(p^*) - f(p_*)]$$

where  $\kappa = \kappa(q, k^* - k, k - k_*)$ ,  $k_* = 2np_* + 1$ ,  $k = \lceil 2np \rceil$  and  $k^* = \lfloor 2np^* - 1 \rfloor$ .

b.  $\forall \varepsilon \in (0, p^* - p_*)$ ,  $\forall z \in N$  and  $\forall G \in N$  there exists  $n^* \in N$  such that for all  $n \geq n^*$

$$\{zG, z; p_*, p^* - p_*\} \succeq \{zG; p^* - \varepsilon\}$$

What does part (a) tell? Suppose that for some interval  $(p_*, p^*) \subset [0, 1]$ , the three outcome lottery,  $S_i$  is weakly preferred to the two outcome lottery  $R_i$  for all  $p_i = i/2n \in (p_*, p^*)$  for some  $n \in N$ . Then statement (\*) offers an upper bound on the difference between the transformed probability of any given  $p \in (p_*, p^*)$  and  $p_*$ . Note that the smaller the upper bound the more flat the transformation of the probabilities, hence the more implausible the implication with respect to risk avoiding behavior. This leads to the result stated in part (b).

Part (b) of the Theorem says that no matter how large the multiplier  $G$  is for sufficiently large  $n$  (that is for prospects  $S_i$  sufficiently close to the hypotenuse) weak preference for the safer prospect  $S_i$  for  $p_i \in (p_*, p^*)$  implies that the agent would not be willing to exchange prize  $z$  with probability  $p^* - p_*$  with prize  $zG$  with probability  $p^* - p_* - \varepsilon$  and 0 with probability  $\varepsilon$ .

The following examples provide numerical illustrations of parts (a) and (b) of the theorem for different intervals of probabilities where prospect  $S_i$  is preferred over prospect  $R_i$ .

Table 2 reports further illustrations of probability distortions and paradoxical risk aversion implications for different values of  $n$  and for  $v(h) \geq 3v(m)$ .

**Example 1:** Suppose that  $S_i \succeq R_i$ , for all  $p_i = i/100 \in (0.5, 1)$ . and  $R_j \succ S_j$ , for some  $p_j = j/100 \in (0, 0.5)$ . In the probability triangle this means that the pattern of risk aversion in the supposition of the Theorem holds for each pair  $i$  above the  $45^\circ$  ray. An application of inequality (\*) with  $p=0.75$  and  $p_* = 0.5$  reveals that  $f(0.75) - f(0.5) \leq [1 - f(0.5)]/K(q, 24, 24)$ . If prizes  $h$  and  $m$  are such that  $v(h) \geq 2.5v(m)$  then one has  $f(0.75) - f(0.5) < 0.00004[1 - f(0.5)]$ . What are risk avoiding implications of a probability transformation that is this flat on the interval  $[0.5, 0.75]$ ? Our individual strictly prefers a lottery that offers prizes \$20 and \$500,000 with equal probability to the lottery that offers \$500,000 with probability 0.75 and \$0 otherwise.<sup>6</sup> Thus, our individual would be unwilling to trade-off \$20 with probability 0.5 in exchange for \$500,000 with 0.25 and 0 with probability 0.25, which is an implausible risk aversion.

**Example 2:** Suppose that  $S_i \succeq R_i$ , for all  $p_i = i/100 \in (0, 0.5)$ , and that  $R_j \succ S_j$ , for some  $p_j = j/100 \in (0.5, 1)$ . In the probability triangle  $i$ - pairs of prospects that satisfy the supposition in the Theorem are located below the  $45^\circ$  line. For  $p=0.25$  a direct application of statement (\*) with  $v(h) \geq 2.5v(m)$  reveals that  $f(0.25) < 0.00004f(0.5)$ , again a severe probability distortions is implied. The last inequality implies preference for a prospect that pays \$100 or \$0 each with probability 0.5 over a prospect that offers \$2.5 million or \$0 with probabilities 0.25 and 0.75 respectively.

**Example 3:** Suppose that  $S_i \succeq R_i$ , for all  $p_i = i/100 \in (0.2, 0.8)$ , and that  $R_j \succ S_j$ , for some  $p_j = j/100 \in (0, 0.2) \cup (0.8, 1)$ . In terms of the example in section 2 our individual weakly

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<sup>6</sup> Indeed, let 0.5M denote \$500,000 and use subadditivity of  $v(\cdot)$  and the upper bound on  $f(0.75) - f(0.5)$  to verify that  $v(0.5M)(f(0.75) - f(0.5)) \leq 25000v(20)(f(0.75) - f(0.5)) < v(20)(1 - f(0.5))$ ; Rearrange terms in the last inequality to get  $v(0.5M)f(0.75) < v(0.5M)f(0.5) + v(20)(1 - f(0.5))$ .



prefers having two white balls in the urn over replacing them with one black (increasing this way the number of black balls by one more) and one green ball (increasing this way the number of green balls by one more); he has this preference whenever the ball composition in urns contains at least 20 green balls and at least 20 black balls. An implication of this preference for rank dependence models is preference for a prospect that pays \$1.9 million with probability 0.2, \$10 with probability 0.6 or \$0 with probability 0.2 over a prospect that pays \$1.9 million or \$0 each with probability 0.5. So, our agent would not be willing to exchange \$10 with probability 0.6 for increasing the probability of getting \$1.9 million (from 0.2 to 0.5) because that exchange increases the probability of prize \$0 (from 0.2 to 0.5).

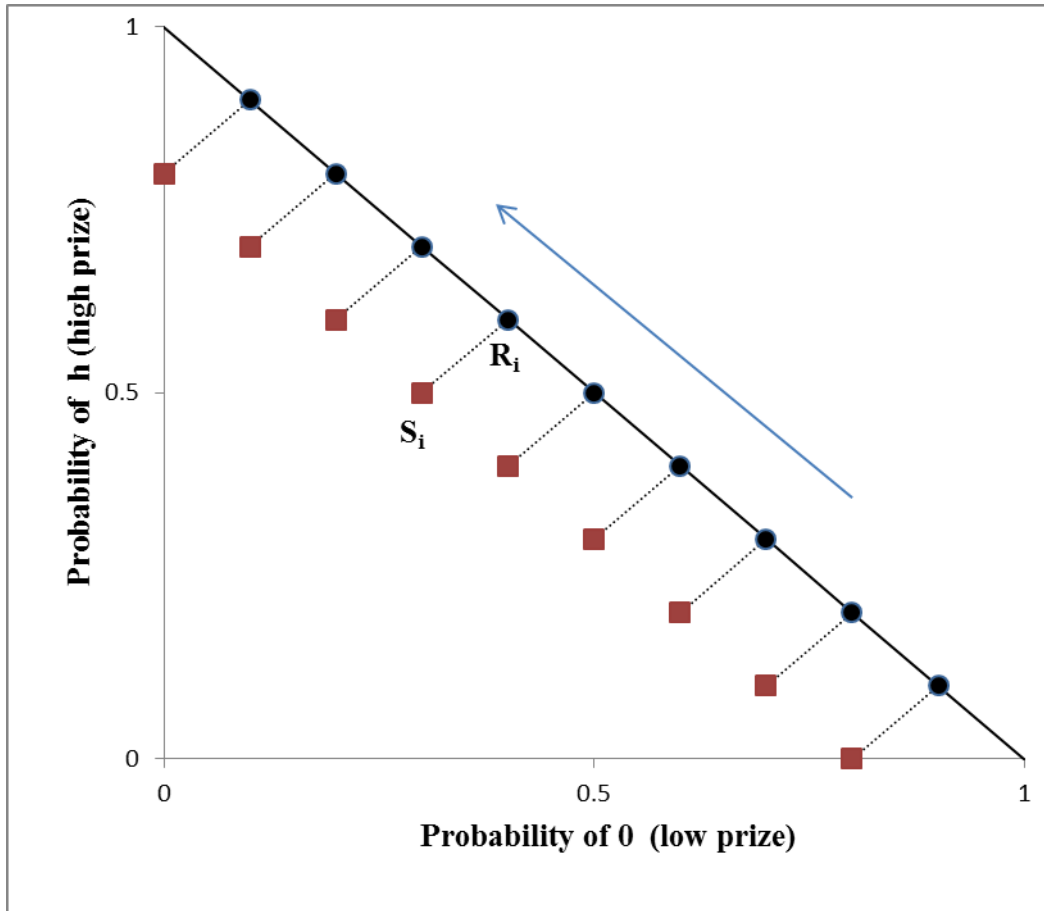
Table 2 (bottom part) shows other numerical illustrations of paradoxical risk aversion. For example, preference for the three outcome lottery for all  $p_i = i / 20 \in (0,1)$  implies that \$100 for sure is preferred to 34 million with probability 0.1 or 0 otherwise (see row n=10, p=0.1 column in Table 2). Or preference for the three outcome lottery for all  $p_i = i / 50 \in (0,0.5)$  implies that the even odds prospect with prizes \$100 or \$0 is preferred to 0.10 billion with probability 0.1 or 0 otherwise (see row n=25, most right G column in Table 2).

#### **4. Implausibility of Modeling Risk Aversion as Attitude towards Probabilities**

Previous literature has focused on the inability of expected utility theory (EUT) to rationalize some postulated patterns of choices. Allais (1953) introduced patterns of choices under risk that (if observed) refute expected utility. Allais' critique was directed at the linearity in probabilities property of the EUT functional. Rabin (2000) patterns of risk aversion were directed at the nonlinearity in payoffs property of the EUT functional; they call into question the ability of the expected utility of terminal wealth model to rationalize risk aversion at large stakes and at small stakes.

This paper introduces a pattern of risk aversion that if explained by probabilistic risk attitudes implies implausible risk aversion. It is shown how attributing risk aversion to attitude towards small probabilities can produce extreme probability distortions that have paradoxical implications. Yaari's dual theory of expected utility, Rank Dependent Utility and Cumulative Prospect Theory, that are not vulnerable to the Allais paradox are vulnerable to the paradoxical risk aversion that follows from nonlinear transformation of probabilities, the very property that makes them accommodate the Allais paradox. This paper argues that rank dependent utility

models have problems rationalizing both risk aversion behavior for small and large probabilities. This is true whether the size of payoffs is small, like in most laboratory experiments, or large.



**Figure 1.** Illustration of pairs of lotteries ( $S_i$ ,  $R_i$ ) in the probability triangle

**Table 2. Numerical Illustrations of Implications of Risk Aversion Pattern in the supposition of the Theorem for  $v(h)/v(m) \geq 3$**

<b>Distortions of Probabilities</b>					
$k_*/2n=1/2n$	$k^*/2n=1-1/2n$			$k^*/2n=0.5-1/2n$	
<b><u>n</u></b>	$k^*/2n$	$f(0.1) <$	$f(0.5) <$	$k^*/2n$	$f(0.1) <$
5	0.9	0.00098	0.03031	0.4	0.03226 $f(0.5)$
10	0.95	$0.29 \times 10^{-5}$	0.00098	0.45	0.00294 $f(0.5)$
25	0.98	$0.28 \times 10^{-13}$	$0.30 \times 10^{-7}$	0.48	$0.93 \times 10^{-6} f(0.5)$
50	0.99	$0.81 \times 10^{-27}$	$0.89 \times 10^{-15}$	0.49	$0.91 \times 10^{-12} f(0.5)$
<b>Paradoxical Risk Aversion</b>					
$k_*/2n=1/2n$	$100 \succ \{G, 0; p, 1-p\}$			$\{100, 0; 0.5, 0.5\} \succ \{G, 0; 0.1, 0.9\}$	
<b><u>n</u></b>	$k^*/2n$	$p = 0.1$ <b>G</b>	$p = 0.5$ <b>G</b>	$k^*/2n$	<b>G</b>
5	0.9	102,000	3,300	0.4	3,100
10	0.95	$0.34 \times 10^8$	102,000	0.45	34,100
25	0.98	$0.36 \times 10^{16}$	$0.33 \times 10^{10}$	0.48	$0.10 \times 10^9$
50	0.99	$0.12 \times 10^{30}$	$0.11 \times 10^{18}$	0.49	$0.11 \times 10^{15}$

### Appendix: Proof of the Theorem

We first prove one proposition and one Lemma. Then we use these two results to prove the main theorem.

Let set  $\Psi(k_*, k^*) = \{k_*/2n, (k_*+1)/2n, \dots, k^*/2n\}$  be defined for any given two integers  $k_*, k^* \in N$  such that  $1 \leq k_* < k^* \leq 2n-1$ .

**Proposition A.1:** Let prizes  $h > m > 0$  and  $n \in N$  be given. Suppose that  $q > 1$  and  $S_i \succeq R_i$ , for all  $p_i \in \Psi(k_*, k^*)$ . Then  $\forall k \in \Psi(k_*, k^*)$

$$(a.1) \quad f\left(\frac{k}{2n}\right) - f\left(\frac{k_*-1}{2n}\right) \leq \frac{1}{\kappa} \left[ f\left(\frac{k^*+1}{2n}\right) - f\left(\frac{k_*-1}{2n}\right) \right]$$

where  $\kappa = \kappa(q, k^* - k, k - k_*)$ .

**Proof.** If  $S_i \succeq R_i$  for some  $i \in \{k_*, \dots, k^*\}$  then according to statement (2) the agent has revealed

$$(a.2) \quad v(h)f\left(\frac{i-1}{2n}\right) + v(m) \left( f\left(\frac{i+1}{2n}\right) - f\left(\frac{i-1}{2n}\right) \right) \geq v(h)f\left(\frac{i}{2n}\right), \quad i = k_*, \dots, k^*$$

Adding and subtracting  $v(m)f(i/2n)$  on the left-hand-side of the above inequality, rearranging terms and using notation  $q = v(h)/v(m) - 1$ , one has

$$(a.3) \quad f\left(\frac{i+1}{2n}\right) - f\left(\frac{i}{2n}\right) \geq q \left( f\left(\frac{i}{2n}\right) - f\left(\frac{i-1}{2n}\right) \right), \quad i = k_*, \dots, k^*$$

If  $S_j \succeq R_j$  for all  $j = i, \dots, i+t$ ,  $i, i+t \in \{k_*, \dots, k^*\}$  then application of inequality (a.3)  $t$  times gives

$$(a.4) \quad f\left(\frac{i+t+1}{2n}\right) - f\left(\frac{i+t}{2n}\right) \geq q \left( f\left(\frac{i+t}{2n}\right) - f\left(\frac{i+t-1}{2n}\right) \right) \geq \dots \geq q^{t+1} \left( f\left(\frac{i}{2n}\right) - f\left(\frac{i-1}{2n}\right) \right)$$

To show statement (a.1) and complete the proof it suffices to show that  $\forall k \in \{k_*, \dots, k^*\}$ ,

$$(a.5) \quad f\left(\frac{k}{2n}\right) - f\left(\frac{k_*-1}{2n}\right) \leq \left( f\left(\frac{k}{2n}\right) - f\left(\frac{k-1}{2n}\right) \right) \sum_{j=0}^{k-k_*} q^{-j}$$

and

$$(a.6) \quad f\left(\frac{k^*+1}{2n}\right) - f\left(\frac{k}{2n}\right) \geq \left(f\left(\frac{k}{2n}\right) - f\left(\frac{k-1}{2n}\right)\right) \sum_{j=0}^{k^*-k} q^{j+1}$$

because the last two inequalities imply that

$$\frac{1}{\sum_{j=0}^{k-k_*} q^{-j}} \left(f\left(\frac{k}{2n}\right) - f\left(\frac{k_*-1}{2n}\right)\right) \leq \frac{1}{\sum_{j=0}^{k^*-k} q^{j+1}} \left(f\left(\frac{k^*+1}{2n}\right) - f\left(\frac{k}{2n}\right)\right);$$

rearrange terms and use notation  $\kappa = 1 + \sum_{j=0}^{k^*-k} q^{j+1} / \sum_{j=0}^{k-k_*} q^{-j}$  to obtain statement (a.1)

$$f\left(\frac{k}{2n}\right) - f\left(\frac{k_*-1}{2n}\right) \leq \frac{1}{\kappa} \left[f\left(\frac{k^*+1}{2n}\right) - f\left(\frac{k_*-1}{2n}\right)\right].$$

Inequality (a.5) follows directly from inequality (a.4) and some rearrangement of terms

$$f\left(\frac{k}{2n}\right) - f\left(\frac{k_*-1}{2n}\right) = \sum_{i=k_*}^k \left(f\left(\frac{i}{2n}\right) - f\left(\frac{i-1}{2n}\right)\right) \leq \left(f\left(\frac{k}{2n}\right) - f\left(\frac{k-1}{2n}\right)\right) \sum_{j=0}^{k-k_*} q^{-j}$$

Similarly, for inequality (a.6) verify that

$$f\left(\frac{k^*+1}{2n}\right) - f\left(\frac{k}{2n}\right) = \sum_{i=k+1}^{k^*+1} \left(f\left(\frac{i}{2n}\right) - f\left(\frac{i-1}{2n}\right)\right) \geq \left(f\left(\frac{k}{2n}\right) - f\left(\frac{k-1}{2n}\right)\right) \sum_{j=0}^{k^*-k} q^{j+1}$$

**Lemma A.1.** If  $q > 1$  then  $\lim_{t \rightarrow \infty} \kappa(q, t, s) = \infty$

**Proof.** It follows from  $\kappa(q, t, s) = 1 + \sum_{j=0}^t q^{j+1} / \sum_{i=0}^s q^{-i} = \frac{q^{2+t+s} - 1}{q^{1+s} - 1} > q^{1+t}$

### Proof of the Theorem

**Part a.** Suppose that  $S_i \succeq R_i$ , for all  $p_i = i/2n \in (p_*, p^*)$  for some  $n \in N$ . Let  $p \in (p_*, p^*)$  be given. Without any loss of generality let  $p_*$  be written as  $p_* = (k_* - 1)/2n$  for some  $k_* \in N$ . Let  $k$  denote the smallest integer larger than  $2np$ , that is  $k = \lceil 2np \rceil$  and  $k^*$  denote the largest integer smaller than  $2np^* - 1$ , that is  $k^* = \lfloor 2np^* - 1 \rfloor$ . By the supposition in the theorem and the construction of  $k, k_*, k^*$  one has  $S_i \succeq R_i$ , for all  $p_i \in \Psi(k_*, k^*)$ . By Proposition A.1 and construction one has

$$f(p) - f(p_*) \leq f\left(\frac{k}{2n}\right) - f\left(\frac{k_* - 1}{2n}\right) \leq \frac{1}{\kappa} \left[ f\left(\frac{k_* + 1}{2n}\right) - f\left(\frac{k_* - 1}{2n}\right) \right] \leq \frac{1}{\kappa} [f(p^*) - f(p_*)]$$

where  $\kappa = \kappa(q, k^* - k, k - k_*)$ , which completes the proof.

**Part b.** For any given  $\varepsilon \in (0, p^* - p_*)$ ,  $G \in \mathbb{N}$  and  $q > 1$  take  $n^* \geq (1 + \ln G / \ln q^2) / \varepsilon$ . Let  $n \geq n^*$  be given and suppose that  $S_i \succeq R_i$ , for all  $p_i = i / 2n \in (p_*, p^*)$ . As above construct  $k = \lceil 2np \rceil$ ,  $p = p^* - \varepsilon$  and  $k^* = \lfloor 2np^* - 1 \rfloor$ , and apply Lemma A.1 and  $n \geq n^*$  to get

$$\kappa(q, k^* - k, k - 1) \geq q^{k^* - k + 1} \geq q^{2(n\varepsilon - 1)} > G.$$

Next applying subadditivity of  $v(\cdot)$ , the last inequality and inequality (\*) in part (a) of the theorem one has

$$\begin{aligned} v(zG)(f(p^* - \varepsilon) - f(p_*)) &\leq Gv(z)(f(p^* - \varepsilon) - f(p_*)) \leq \kappa v(z)(f(p^* - \varepsilon) - f(p_*)) \\ &\leq v(z) [f(p^*) - f(p_*)] \end{aligned}$$

Hence,  $v(zG)f(p^* - \varepsilon) \leq v(zG)f(p_*) + v(z) [f(p^*) - f(p_*)]$ , which completes the proof

$$\{zG, z; p_*, p^* - p_*\} \succeq \{zG; p^* - \varepsilon\}.$$

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