Price as a choice under nonstochastic randomness in finance[1]

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Abstract Closed sets of finitely-additive probabilities are statistical laws of statistically unstable random phenomena. Decision theory, adapted to such random phenomena, is applied to the problem of valuation of European options. Embedding of the Arrow-Debreu state preference approach to options pricing into decision theoretical framework is achieved by means of considering option prices as decision variables. A version of indifference pricing relation is proposed that extends classical relations for European contingent claims to statistically unstable random behavior of the underlying. A static hedge is proposed that can be called either the model specification hedge or the uncertainty hedge or the generalized Black-Scholes delta. The obtained structure happens to be a convenient way to address such traditional problems of mathematical finance as derivatives valuation in incomplete markets, portfolio choice and market microstructure modeling.

Key words Statistical instability, Randomness, Finitely-additive measures, Decision theory, Uncertainty profiling, Derivatives Valuation, Portfolio choice, Bid-Ask Spread

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1. Introduction

The goal of this paper is two-fold. First of all, we address the issue of reconciliation of the Arrow-Debreu state preference approach to derivatives valuation [1] with post-Savagian decision theory, concerned with the attitude toward uncertainty and requiring existence of sets of finitely-additive probabilities [27], [28], [32], [21], [13], [35]. Embedding of the Arrow-Debreu securities market model into decision theoretical structure with arbitrary (in particular, uncountable) state space is achieved by considering option prices, along with options pay-offs and positions, as decision variables and by identifying profits as decision consequences and as utilities. Applying a form of indifference argument we obtain pricing relations for bounded pay-offs as well as an extension of classical Black-Scholes delta. In particular, the put-call parity pricing argument [18] can be seen as a particular case of this more general version of indifference valuation, where indifference is considered relatively to a rationally class.[5] The resulting structure happens to be a convenient way to address derivatives valuation in incomplete markets as well as an opportunity to introduce in mathematical finance the alternative interpretation of the sets of finitely-additive probabilities as statistical laws of statistically unstable random phenomena [29], [30], [31]. Let us commence with a few words about this interpretation and then return in more details to the first issue.

Countably-additive probability measures are the common tool for modeling of randomness in mathematics, in general, and in mathematical finance, in particular [6], [7], [17], [20], [22], [52]. Notwithstanding the well known reasons for this situation and the technical benefits it entails, let us focus on the two features that seem to us the drawbacks of this approach.

First, it is known, that sometimes countable-additivity is too restrictive in order to characterize the absence of arbitrage opportunity and the weaker property of finite-additivity is required instead [5]. At the same time, it is known that the most general definition of the absence of arbitrage opportunities does not require countable additivity of probability measure and that finite-additivity is already sufficient [55].[6]

The second feature is related to the fact that those phenomena in the financial industry that are commonly considered as random, and that are modeled today by means of countably additive probabilities, often happen to be at the same time statistically unstable [38]. Attempts to extend the possibilities or to avoid the contradictions of stochastic modeling in finance are

[5] As is often the case with papers addressing indifference valuation, we do not consider here the no-arbitrage setting.

[6] See, for instance, [57] for a concise introduction to finitely-additive measures.

numerous [3], [9], [46], [19]. Here it is reasonable to quote A.N. Kolmogorov [36]:”Speaking of randomness in the ordinary sense of this word, we mean those phenomena in which we do not find regularities allowing us to predict their behavior. Generally speaking, there are no reasons to assume that random in this sense phenomena are subject to some probabilistic laws. Hence, it is necessary to distinguish between randomness in this *broad sense* and stochastic randomness (which is the subject of probability theory).” One can find similar views in E. Borel [10]. Therefore it is natural to classify statistically unstable random phenomena of the financial industry as a subclass of random in broad sense phenomena. But do statistically unstable, or “nonstochastic”[7] random phenomena follow any statistical law?

The positive answer to these questions was given first in [29], and then in [30]. Namely, it was proved that any random phenomenon, modeled by means of so called *sampling nets*, follows some statistical law in the form of a closed in *-weak topology family of finitely additive probabilities, called *statistical regularity*, and vice versa, any statistical regularity is a statistical regularity of a certain sampling net.[8] This result will be referred to hereunder as the theorem of existence of statistical regularities [31]. In light of Kolmogorov’s statement quoted above, it suggests that, to describe random phenomena with some generality, statistical regularities are more appropriate tools than single countably–additive probability measures.[9] In other words, statistical regularities extend the means of mathematical description of randomness, while this description mainly rests today on stochastic processes. Yet, suspicion that this current practice underestimates randomness and risks related to financial markets is today commonplace.

Closed sets of finitely-additive measures emerge also as a consequence of axioms of rational choice in several theorems of decision theory [27], [28], [32], [21], [13], [35]. Due to the theorem of existence of statistical regularities, the closed sets of finitely-additive probabilities acquire an interpretation different from the one traditionally attributed to them in decision theory (imprecise probabilities, non-unique priors, and robust statistics). In what

[7] The expression appeared in [54] in the context of Kolmogorov’s complexity, meaning “more complex than stochastic”. We use it here as a synonym of “statistically unstable”, following [29] and [31].
[8] The words *statistical regularity* should be understood as *statistical law* or *statistical pattern.* At the same time, *net* is a synonym of *directedness* and of *generalized sequence*
[9] An illustration of this can be found in financial markets, where a single underlying is characterized by a family of probability distributions, represented by the implied volatility surface. Traditionally, this phenomenon is appreciated in terms of different subjective probability distributions of market participants concerning the random behavior of the underlying. In the spirit of stochastic modeling, complex stochastic processes (with local or stochastic volatility) are designed in order to reproduce this phenomenon. The concept of statistical regularity tells that it is *normal* for a random phenomenon to be characterized by a family of finitely-additive probability distributions.

follows we shall remain within this new interpretation and our main source will be the first two references in the above list.[10]

In order to create the link between decision theory and mathematical finance, we embed the Arrow-Debreu state preference approach to option pricing into the general decision theoretical structure. We achieve this, first of all, by means of considering securities prices as decision variables. In this way the problem of option pricing becomes the decision making problem. In particular, unlike the standard Arrow-Debreu model, where the state space is finite, we consider an arbitrary state space (in particular, uncountable). The form of decision theory we use, together with the notion of statistical regularities, allows for a separation between the shape of the utility function and the risk attitude of decision maker. Risk and uncertainty attitude are embedded in the axiomatic description of the decision maker's rationality class. Using profits as (linear) utility functions, we show that the standard put-call parity argument, known as well as the static replication argument [18], can be put into a formal framework of decision theory and presented as an *indifference* argument [12], where indifference is considered relatively to a rationality class. As a result, we obtain an extension of the Black-Scholes formula for European options to the more general case of some nonstochastic random evolution of the stock price. Besides, we provide similar extensions for the forward and spot stock prices. Indifference prices happen to be identical for uncertainty avers and uncertainty prone decision makers. We propose a specific expression for the hedge ratio that may be thought of as a generalization of the classical Black-Scholes delta hedge. However, it may be thought of as hedging also the risk of misspecifying the formal model, or as a hedge against this type of uncertainty.

The article is organized as follows. In Section 2.1 we recall the terminology and the set-up of decision systems formalism. In Section 2.2 we explore structural similarities between the agent-independent elements of the Arrow-Debreu model of securities market under uncertainty, following in this Avellaneda [4], and the canonical representations of decision problems (so called decision schemes). In Section 2.3 we define the indifference price for

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[10] Unlike other works in this list, the sets of finitely additive probability measures appearing in the first two papers are not necessarily convex. This makes a direct link to statistical regularities. The convexity constraint present in other works, while being of seemingly little importance in decision –making-context (due to the well known extremal properties of linear functionals), at the same time unduly narrows the spectrum of random phenomena. A detailed discussion of the reasons to this distinction belongs to decision theory and is beyond the scope of the present article. Such a discussion can be found, for instance, in [31] and [43], [44], [45]. Here we just comment, that the system of axioms to which we refer, apart from being the first work providing decision-theoretical justification of statistical regularities, is free of any traces of information about the unknown and is formulated for arbitrary decision sets.

financial instruments with general pay-offs. Then (Section 2.4-2.5) we obtain extensions of some classical results of mathematical finance, namely the forward price and the Black-Scholes formula for European options to the more general case of nonstochastic random behavior of the underlying. In Section 2.6 an expression of the static hedge already mentioned above is proposed. Appendix is dedicated to the examples of manifestations of uncertainty attitude in portfolio selection and bid-ask pricing problems. In the Appendix A we demonstrate that in what concerns portfolio choice, uncertainty aversion and uncertainty propensity, two distinct psychological features defined axiomatically, manifest themselves as the choices of respectively diversified and directional portfolios, even when utility functions are linear. In Appendix B we show that uncertainty averse and uncertainty prone decision makers are likely to respectively rise or lower their bid-ask spreads in situations of uncertainty. This provides the rationale to introduce a speculative component of the bid-ask spread and call it the uncertainty price. A discussion around some related topics concludes the paper.

2. Price as a decision variable under nonstochastic randomness

2.1. Basic features of the decision systems formalism

In order to systematically apply decision theoretical approach to problematic of mathematical finance, throughout this article we shall use the terminology related to the formalism of *decision system* [31]. Let us try to recall briefly some basics of decision theory, in general, and of this formalism, in particular.

Let there be a triplet

$$Z = (\Delta, \Theta, L), \tag{1}$$

where $\Delta, \Theta -$ are *arbitrary* sets and $L: \Delta \times \Theta \to \mathbb{R}$ is a real bounded function. These elements are endowed with the following meaning: $\Delta -$ is the set of decisions or acts, $\Theta -$ is the set of values of the unknown parameter, also called the state of Nature, $L -$has the meaning of a utility or a loss function. The set of values of function $L(\cdot,\cdot)$ constitutes the set $C \subseteq \mathbb{R}$ of consequences. A decision maker, without knowing in advance which $\theta \in \Theta$ will take place, chooses a decision $d \in \Delta$, that leads her to one of the possible consequences $L(\theta, d)$, $\theta \in \Theta$. For instance, in finance the values of function $L(\theta, d)$ may be the profits of the operation d. The triplet (1) is called a *matrix decision scheme*. Though this triplet is clearly the canonical representation of decision problems [15], [34], [37], [50], distinguishing it as the matrix decision scheme is important in the framework of the formalism of decision systems.

A decision maker is characterized by the preference relation β_C on consequences (for instance, higher values of profits are better than lower). In order to make a justifiable decision, a decision maker must have a preference relation β_Δ, or a criterion, on the set of decisions that would allow her to compare decisions. Provided the preference relation on consequences, β_C, is fixed, if the mapping, satisfying some natural consistency conditions, of the preference relation β_C on the set of possible preference relations B_Δ on Δ is not single-valued, then it is said that *uncertainty* exists in the decision scheme Z (more rigorous definitions appear in [31], pp. 30-33)[11]. A system of axioms describing, in particular, decision maker's behavior under conditions of uncertainty, puts some β_Δ in correspondence to a given β_C. Such a system of axioms is called a *projector*, or a *criterion choice rule,* $\pi : (Z, \beta_C) \rightarrow \mathrm{B}_\Delta$. Each criterion choice rule π describes a certain *rationality class*, or a class of decision makers. The pair consisting in a decision scheme, $Z,$ and in a criterion choice rule, π, is called a *decision system* $T = (Z, \pi)$.[12]

According to fundamental theorems of decision theory, there exists a one to one correspondence between a rationality class and a specific functional form, the decision *criterion* (for example expected utility), representing the preference relation β_Δ. We shall denote by Π_1 the class of decision makers which is characterized by the uncertainty aversion axiom, called also the guaranteed result principle generalized for mass events [27, 28, 30]. For that class $\Pi_1,$ the preference relation β_Δ on decisions is characterized by the following criterion

$$L_Z^*(d) = \max_{p \in P} \int L(\theta, d) p(d\theta), \tag{2}$$

where $L(\cdot,\cdot)$ has the meaning of some loss function, and

$$L_Z^*(d) = \min_{p \in P} \int L(\theta, d) p(d\theta), \tag{3}$$

where $L(\cdot,\cdot)$ has the meaning of some utility function, and where $P(\theta)$ is a *statistical regularity*, as mentioned in the introduction. This statistical regularity may have objective or

[11] This definition of uncertainty differs from the one traditionally used in decision sciences. It is a stronger concept, for it does not only refer to situations where one is not sure about which distribution applies, but to situation where it is not even known whether any distribution applies: it may be that a statistical regularity applies, but not necessarily a distribution, the latter being a countably-additive measure. It thus is a richer notion than the concept of "ambiguity", falling short of being linked to "Hicksian" uncertainty. The concept referred to in this paper might help qualifying "sources" of uncertainty [54], but this is beyond the scope of this paper. It is closely related to the existence of so called uncertainty functions [16] and uncertainty measures [31] used in information and control theory.

[12] If there exists some complementary information I about θ (like "θ is random with statistical regularity P"), then the pair S=(Z, I) is called a *model of decision situation* and the pair $T = (S, \pi)$ is called *decision system*.

subjective origins, describing in both cases some random in a broad sense (or nonstochastic) behavior of the unknown parameter $\theta \in \Theta$. Given a statistical regularity $P(\Theta)$ is constructed, the decision maker will order the decision set according to the values taken by the criterion. Criteria (2)-(3) are correspondingly a convex and a concave functional on the set of decisions Δ. Furthermore, it was shown that when the uncertainty aversion axiom is replaced by an *uncertainty propensity axiom*, then criteria (2)-(3) change the convexity sign [44], [45].[13] The class of the uncertainty prone decision makers will be denoted as Π_2.[14] It is reasonable to call the axiom describing the attitude toward uncertainty within a given rationality class as the *uncertainty profile,* stressing that this is one of the core psychological features of a decision maker. Further in this paper, we propose an interpretation of different uncertainty profiles in terms of preferences of portfolio choice. Note, however, that some results of this article are invariant for classes Π_1 and Π_2.

Finally, for brevity' sake, we propose the following notations.

Definition 1. Let $f(\theta)$ be a bounded real function, $f\colon \Theta \to \mathbb{R}$, and P a statistical regularity on Θ. Denote

$$\langle f\rangle_P = \frac{\min\limits_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta) + \max\limits_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta)}{2}$$

and call it the *statistical mean* of the function *f* with respect to statistical regularity *P*. Denote

$$[f]_P = \max_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta) - \min_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta)$$

and call it the *statistical variation* of the function *f* with respect to statistical regularity *P*.

2.2. Decision scheme as a model of securities market under uncertainty

Recall the one-period Arrow-Debreu model of securities market under uncertainty [1]. Following Avellaneda [4] (pp 2-3), throughout this paper we shall understand this model in a narrow sense, meaning only those elements of this model that do not depend on the agent. Let Θ be the set of states of market θ_j, $j = 1, \dots, M$ and let F be the cash-flow matrix $\|f_{ij}\|$ of $i = 1, \dots, N$ securities in each state j of the market. Let $\Delta = \mathbb{R}^N$ be the set of financial

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[13] Similar results for convex P were obtained in [13] and extended, though in somewhat different setting, in [35]

[14] When the decision maker is characterized by uncertainty neutrality feature, then convexity of the criterion disappears and $L_Z^*(d) = \int L(\theta, d)p(d\theta)$, where p is a single finitely-additive probability measure on Θ (see [43]).The class of uncertainty neutral decision makers will be denoted as Π_3.

positions $d = (d_1, \dots, d_N) \in \Delta$ and $U = \mathbb{R}^N$ be the set of prices $u = (u_1, \dots, u_N) \in U$ corresponding to each of the N securities. There are only two values of time in this model: $t = 0$, when positions d are taken and the amount $u_d = u \times d = \sum_{i=1}^{N} u_i d_i$ payed, and $t = T$, when the state of market $\theta_j, j = 1, \dots, M$, unknown at the time moment $t = 0$, is revealed and the cash-flow $F_j^d = d \times F_{\cdot,j} = \sum_{i=1}^{N} u_i f_{ij}$ collected.

This model can be thought of in terms of one-step decision problem. Indeed, let the decision scheme $Z = (\Delta, \Theta, L)$ is given, where

$$\Delta \equiv D^{(\infty)} = \bigcup_{n=1}^{\infty} D^n, \quad D^n = \underbrace{D \times D \times \dots \times D}_{n}, \quad D = F \times U \times Q, \tag{4}$$

where Θ is an arbitrary set interpreted as the set of states of the market, F –is a set of bounded real functions on Θ, $F = \{f: \Theta \to \mathbb{R}\}$, that we interpret as pay-offs of financial instruments contingent on the state of the market $\theta \in \Theta$; U –is the set of their prices, that we consider coinciding with the set of real numbers, $U = \{u, u \in \mathbb{R}\}$; Q –is the set of positions, that we as well consider coinciding with the set of real numbers, $Q = \{q, q \in \mathbb{R}\}$, so that $q > 0$ corresponds to a long position, etc. Thus a choice of decision $d \in \Delta$ means the choice of securities, their prices and their amounts or positions. This decision is made at the time moment $t = 0$. At the future time moment $t = T$, the state of market, or the value of the unknown parameter, $\theta \in \Theta$ is observed. The set Θ has in our case the following structure:

$$\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_N, \tag{5}$$

where Θ_i is the set of values of the underlying i , $i = 1,2, \dots, N$. If the underlying i is a stock, then the set $\Theta_i = [0, \infty)$ is the set of prices of this stock at time $t = T$. If the underlying i is a risky fixed income instrument, then $\Theta_i = [0,1]$.

The consequence $L(d, \theta)$, observed at the time moment $t = T$, of the decision $d \in \Delta$, when the state of the market or the value of the unknown parameter is $\theta \in \Theta$, will be

$$L(d, \theta) = \sum_{i=!}^{N_d} q_i\big(-u_i + f_i(\theta)\big) \tag{6}$$

where $d = \left(d_1, d_2, \dots, d_{N_d}\right) \in D^{N_d} = \underbrace{D \times D \times \dots \times D}_{N_d} \in D^{(\infty)} \equiv \Delta$ and where D is given by (4).

Traditionally, when in the Arrow-Debreu model the number M of the states of market is greater than the rank of the cash-flow matrix F, the market is usually called incomplete (see details in [4]). Hence, decision scheme Z defined by (4) - (6) is by construction a model of an incomplete market.

Consider some examples. Let the decision at the time moment $t = 0$ be a long position in the stock *i*. Then $d \in D, d = (f, u, q), f(\theta) = \theta_i \in \Theta_i = [0, \infty), u$ −is the bid price of the stock, q −is the quantity of the stocks purchased and $L(d, \theta) = q(-u + \theta_i)$, $\theta = (\theta_1, \dots, \theta_i, \dots, \theta_N) \in \Theta = \Theta_1 \times \dots \times \Theta_\mathrm{i} \times \dots \times \Theta_N$. In what follows we shall consider that all purchase of securities are financed via a bank loan with interest rate *r*, thus leading to a repayment of e^{rT} and all proceeds from sales of securities are placed as bank deposits at the same interest rate, thus yielding proceeds of e^{rT}. That is if the purchase of q units of security $f(\theta)$ at the price *u* is financed by the bank loan of *qu* money units, then $d \in D^2 = D^1 \times D^1, d = (d_1, d_2), d_1 = (f, u, q), d_2 = (e^{rT}, 1, -qu)$ and the consequences are

$$L(d, \theta) = \sum_{i=!}^{2} q_i\big(-u_i + f_i(\theta)\big) =$$

$$q\big(-u + f(\theta)\big) - qu(-1 + e^{rT}) = q\big(-ue^{rT} + f(\theta)\big). \tag{7}$$

The resulting expression (7) means that the decision maker owes the bank the sum que^{rT} and receives the pay-off $qf(\theta)$ from the seller. In case of sale of the security $f(\theta)$ and deposit of proceeds, the consequences are

$$L(d, \theta) = \sum_{i=!}^{2} q_i\big(-u_i + f_i(\theta)\big) =$$

$$q\big(u - f(\theta)\big) + qu(-1 + e^{rT}) = q\big(ue^{rT} - f(\theta)\big). \tag{7'}$$

The resulting expression (7') means that the decision maker receives from the bank the sum que^{rT} and owes the pay-off $qf(\theta)$ to the buyer. Remark that in this construction the riskless interest rate is a matter of choice as well.

Further, some other terminological parallels can be established. If the situation requiring the choice of decision $d \in \Delta$ is such that the pay-offs, i.e. the elements of the set *F*, are fixed, then the problem of choice of the decision $d \in \Delta$ is equivalent to the choice of securities prices and may be called correspondingly *asset valuation* or *pricing decision problem*. If, on the contrary, the prices of securities are fixed, but securities themselves are not, then the problem of choice of decision $d \in \Delta$ is equivalent to the choice of portfolio of securities and may be called correspondingly *portfolio choice* or *investor's decision problem*. If neither prices, nor securities are fixed, then one is tempted to call the problem of choice of decision $d \in \Delta$ a *market decision problem.* In what follows in this paper we shall be concerned with pricing decision problems. Remark that decision $d \in \Delta$ may be as well called *a transaction.*

In this way decision scheme Z (4-6) becomes an extension of the Arrow-Debreu model of the securities market under uncertainty to arbitrary (in particular, uncountable) state space.

2.3. Indifference price and nonstochastic randomness

According to (4-6), the value $L(\theta, d)$ is the actual profit and loss of the transaction $d \in D$ when the state of market is $\theta \in \Theta$. In this paper we consider linear utility functions of profits, hence coinciding with profits themselves. We admit at first that the decision maker belongs either to the class Π_1 of uncertainty averse decision makers or to the class Π_2 of uncertainty prone decision makers and uses the form (3) of the criterion $L_Z^*(d)$ that has the meaning, correspondingly, of minimal and maximal expected profits related to the decision, or transaction $d \in \Delta$.

If for two decisions $d_1, d_2 \in \Delta$,

$$L_Z^*(d_1) = L_Z^*(d_2) \tag{8}$$

then decision maker is *indifferent* between decisions d_1 and d_2. It is worth noticing that considering the value of the criterion $L_Z^*(d)$ as *the value* in the traditional economic sense[15], we achieve a clear separation of *the value* and *the price*, which is one of the elements of decision d. In particular, for a portfolio of instruments, the prices are additive, while the values are not.

Let the decision maker evaluate the possibilities of purchase and sale of the security with pay-off $f(\theta)$. Then, taking into account (7-7'), the purchase is effectively described by the decision $d_b = (f, u, q)$ and the sale – by decision $d_s = (f, u, -q)$, and the consequences are, correspondingly, $L(d_b, \theta) = q\left(-ue^{rT} + f(\theta)\right)$ and $L(d_s, \theta) = q\left(+ue^{rT} - f(\theta)\right)$. Let this decision maker belongs to the class Π_1. Inserting these expressions in (3) and (8) we obtain

$$\min_{p \in \boldsymbol{P}} \int L(\theta, d_b) p(d\theta) = \min_{p \in \boldsymbol{P}} \int L(\theta, d_s) p(d\theta), \tag{9}$$

or

$$-u\, e^{rT} + \min_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta) = u\, e^{rT} - \max_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta), \tag{10}$$

or, solving for u,

$$u = e^{-rT} \frac{\min_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta) + \max_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta)}{2} = e^{-rT} \langle f \rangle_P. \tag{11}$$

[15]More specifically, $L_Z^*(d)$ is the *subjective* value for the given individual when prices p may be regarded as being market equilibrium ones. Prices are objectively observable, values are subjective and may not be directly observed.

Thus, provided the riskless interest rate is r and the statistical regularity on Θ is $P(\theta)$, the price (11), that has the form of the discounted statistical mean of the pay-off $f(\theta)$ with respect to statistical regularity $P(\theta)$, is the one that makes our uncertainty averse decision maker indifferent between the purchase and the sale of the security $f(\theta)$.[16] The price (11) may be considered as another version of the *indifference price* [12]. It is easy to verify that expression (11) is invariant for the classes of uncertainty aversion, Π_1, and uncertainty proneness, Π_2. In our case, however, a decision maker is indifferent between the role of the buyer and the role of the seller. Traditionally, indifference is required between an investment of a certain amount of money in a risky security and a riskless asset. In our framework, however, the same question would not lead to the above invariance of the price. Indeed, the possible decisions are $d_b = (f, u, 1), d_r = (e^{rT}, 1, u)$ and the consequences are, correspondingly, $L(d_b, \theta) = 1 \cdot (-u + f(\theta))$ and $L(d_r, \theta) = u \cdot (-1 + e^{rT})$. For the class Π_1 one obtains

$$u = e^{-rT} \min_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta), \tag{12}$$

while for the class Π_2 one obtains

$$u = e^{-rT} \max_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta). \tag{13}$$

These relations imply that, facing such an alternative, any uncertainty prone decision maker is ready to pay more for the risky asset than any uncertainty averse decision maker.

Now, as is usually the case with asset valuation, the central question is what statistical regularity $P(\theta)$ does the decision maker use? In principle, this regularity can be a closed family of finitely-additive measures that the decision maker believes to describe the random evolution of the parameter θ.

Traditionally, in mathematical finance, the answer to this question is provided in terms of the absence of arbitrage opportunities and the fundamental theorem of asset pricing [6], [17]. Technically, it is assumed that if market prices are arbitrage-free, then the pricing measure is inferred by means of calibration procedures and extended to all existing pay-offs. The

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[16] In the case when nothing is known about θ save the set of its values, or when P is the set of all finitely-additive probability measures on Θ, including all Dirac delta distributions, (11) becomes $u = e^{-rT} \frac{\min_{\theta \in \boldsymbol{\Theta}} f(\theta) + \max_{\theta \in \boldsymbol{\Theta}} f(\theta)}{2}$. In case the statistical regularity has stochastic character, that is when the set P contains a single stochastic - that is countably-additive - probability measure p, (11) becomes $u = e^{-rT} \int f(\theta) p(d\theta)$.

indifference pricing methodology usually takes the form of expected utility maximization in order to specify the pricing measure [12]. However, additional statistical procedures may be resorted to (as entropy minimization in [49]). A synthesis of the two approaches, based on a dual optimization problem, is an active research topic [8].

Indifference (8) can be used as a requirement as well. When pay-offs are deterministic or risk-free, indifference, if used as a requirement, is analogical to the requirement of the absence of arbitrage opportunities. Similarly to the arbitrage-free hypothesis, one can evoke the indifference hypothesis and interpret market prices in this way. This analogy makes the case of the so called static replication, better known as put-call parity condition, and is known to produce results identical to those obtained in the framework of dynamic replication [18]. We claim that static replication arguments can be formally put into the framework of indifference that is proposed in this article. In the next two sections we show that requiring indifference yields conditions on the "fair" statistical regularity $P(\theta)$ in a sense which will be defined below, first in case of a simple stock and forward contract and then in the case of European options.

2.4. Spot and forward stock prices as decision variables.

Without any loss of generality, we shall consider the case, when $\Theta = \Theta_1 = [0, \infty)$ is the set of prices of a single stock at time $t = T$. Consider the following decision problem. Let the spot stock price θ_0 be a decision in the situation where a decision maker chooses between long and short stock positions held at time $t = T$ and financed with a bank account. Namely, let the decision corresponding to the long position, $d^l \in D$ consist in the bank loan of θ_0 monetary units, purchase of 1 unit of stock at the price θ_0, sale of the stock at the price θ. Then, $d^l = (d_1, d_2), d_1 = (\theta, \theta_0, 1)$, $d_2 = (e^{rT}, 1, \theta_0)$, and according to (7), the profit of this operation is

$$L(d^l, \theta) = -\theta_0 e^{rT} + \theta. \tag{14}$$

Correspondingly, the short-sale of the stock is described by decision $d^s = (d_1, d_2), d_1 = (\theta, \theta_0, -1)$, $d_2 = (e^{rT}, 1, -\theta_0)$, and according to (7'), the profit of this operation is

$$L(d^l, \theta) = +\theta_0 e^{rT} - \theta. \tag{14'}$$

Indifference relation (8) yields,

$$\min_{p \in \boldsymbol{P}} \int (\theta - \theta_0 e^{rT}) p(d\theta) = \min_{p \in \boldsymbol{P}} \int (\theta_0 e^{rT} - \theta) p(d\theta) \tag{15}$$

or

$$\theta_0 = \ e^{-rT} \langle \theta \rangle_P. \tag{16}$$

One can thus conclude that the choice of the spot price θ_0 becomes a condition on statistical regularity P that makes this statistical regularity “fair”.

Now consider as decision the forward price θ_F for the stock supposing that its spot price is θ_0. This means that long forward contract position is described as decision $d_F^l = (d_1, d_2), d_1 = (\theta_F, \theta_0, 1),\ \ d_2 = (e^{rT}, 1, \theta_0)$, and, according to (7), the profit of this operation is

$$L(d^l, \theta) = -\theta_0 e^{rT} + \theta_F. \tag{17}$$

The short forward contract is described by decision $d_F^s = (d_1, d_2), d_1 = (\theta_F, \theta_0, -1)$, $d_2 = (e^{rT}, 1, -\theta_0)$, and according to (7’), the profit of this operation is

$$L(d^l, \theta) = +\theta_0 e^{rT} - \theta_F. \tag{17’}$$

That profits (17) and (17’) do not depend on θ reflect the fact that the pay-off $f(\theta) = \theta_F$ is uniquely defined, or certain. Indifference condition (8) in this case yields

$$\min_{p \in \boldsymbol{P}} \int L(\theta, d_F^l) p(d\theta) = \min_{p \in \boldsymbol{P}} \int L(\theta, d_F^s) p(d\theta)\,. \tag{18}$$

or

$$\theta_F = \theta_0 e^{rT}, \tag{19}$$

which is the classical forward price for a stock the spot price of which is θ_0 (see [26]). Relation (19) reflects the fact that when pay-offs are uniquely defined, indifference condition coincides with the no arbitrage condition.

On the other hand, consider non-deliverable forwards, NDF. In this case, long NDF means the pay-off $f(\theta) = \theta_F - \theta$ at $t = T$ and price $u = 0$ at $t = 0$. Hence the decision describing this transaction is $d_{NFD}^l = d_1 \in D^1, d_1 = (\theta_F - \theta, 0,1)$ and, according to (7), the profit of this operation is

$$L\left(d_{NFD}^l, \theta\right) = -\theta + \theta_F. \tag{20}$$

Decision describing the corresponding short transaction is $d_{NFD}^s = d_1 \in D^1, d_1 = (-\theta_F + \theta, 0,1)$ and, according to (7’), the profit of this operation is

$$L\left(d_{NFD}^l, \theta\right) = +\theta - \theta_F. \tag{20’}$$

The indifference condition (8) yields

$$\theta_F = \ \ \langle\theta\rangle_P. \tag{21}$$

At the same time, from (19) we have

$$\theta_0 = e^{-rT}\ \ \langle\theta\rangle_P. \tag{22}$$

We have just obtained a relation similar to (16). In other words, the problem of choice of the forward contract value is equivalent to the problem of choice of the spot price. Provided the prices θ_0 and θ_F are usually given by the market, equations (21) and (22) become conditions on the "fair" statistical regularity P on Θ**.**

2.5. European options, forward contracts, put-call parity, conditions on *P*.

The pay-off of a European call option is $f(\theta) = (\theta - \theta^*)^+$, where θ – is the underlying stock price at maturity date *T, and* θ^* - is the strike. Hence, from (11) we have

$$u_c = e^{-rT} \langle(\theta - \theta^*)^+\rangle_P. \qquad (23)$$

The pay-off of the corresponding European put is $f(\theta) = (\theta^* - \theta)^+$. Hence

$$u_p = e^{-rT} \langle(\theta - \theta^*)^-\rangle_P. \qquad (24)$$

For the corresponding forward contract on the underlying stock $f(\theta) = \theta - \theta^*$. Hence

$$u_f = e^{-rT} \langle\theta - \theta^*\rangle_P = e^{-rT} (\langle\theta\rangle_P - \theta^*). \qquad (25)$$

Substituting (16), or (22), in (25) one obtains

$$u_f = e^{-rT} (\theta_0 e^{rT} - \theta^*) = e^{-rT} (\theta_F - \theta^*) = \theta_0 - \theta^* e^{-rT}, \qquad (26)$$

where θ_0 – is the spot price, θ_F –is the forward price of the underlying, and θ^* - the strike. This quantity is usually called *the value of the forward contract* [26], but in practical terms it is the market price of the forward contract.

Now it is easy to retrieve the classical put-call parity relation. Consider the following decision scheme. On one hand, a decision maker can buy the call at the price u_c (23), obtaining the bank loan of u_c monetary units with interest rate *r*, requiring a repayment of $u_c e^{rT}$, and simultaneously sell the put for the price u_p (24), placing the proceeds as a bank deposit. This long call-short put transaction is effectively described by decision $d_{cp}^{ls} = (d_1, d_2, d_3, d_4) \in D^4 \in \Delta, d_1 = ((\theta - \theta^*)^+, u_c, 1), d_2 = (e^{rT}, 1, -u_c), d_3 = \big((\theta^* - \theta)^+, u_p, -1\big), d_4 = \big(e^{rT}, 1, u_p\big)$ and accor-ding to (6), the profit of this operation is

$$L\big(d_{cp}^{ls}, \theta\big) = (-u_c + u_p)e^{rT} + (\theta - \theta^*)^+ - (\theta^* - \theta)^+ =$$
$$= \big(-u_c + u_p\big)e^{rT} + (\theta - \theta^*). \qquad (27)$$

On the other hand, the decision maker can take a forward position with the forward price equal to the strike, $\theta_F = \theta^*$, and with the same pay-off as the long call-short put transaction, that is $(\theta - \theta^*)$. The indifference value of this transaction is (26). The decision corresponding to this transaction is $d_F = (d_1, d_2) \in D^2 \in \Delta, d_1 = \big(\theta - \theta^*, u_f, 1\big), d_2 = \big(e^{rT}, 1, -u_f\big)$ and according to (7), the profit of this operation is

$$L(d_F, \theta) = -u_f e^{rT} + (\theta - \theta^*). \qquad (28)$$

Since pay-offs of decisions d_{cp}^{ls} and d_F are the same $f(\theta) = \theta - \theta^*$, it is reasonable to ask whether there could exist some combination of prices that would make the decision maker indifferent between these decisions. Indifference condition (8) yields then:

$$\min_{p \in \boldsymbol{P}} \int L\left(d_{cp}^{ls}, \theta\right) p(d\theta) = \min_{p \in \boldsymbol{P}} \int L(d_F, \theta) p(d\theta)\,, \tag{29}$$

or, taking into account (26) and (27),

$$u_c - u_p = u_f = e^{-rT}\ (\theta_F\ - \theta^*) = \theta_0 - \theta^* e^{-rT}, \tag{30}$$

which is the well known classical put-call parity relation (see, for instance, [26]).

Substituting (23, 24) in (30) one obtains additional conditions on the statistical regularity $P(\theta)$

$$\langle(\theta - \theta^*)^+\rangle_P - \langle(\theta - \theta^*)^-\rangle_P = \theta_0 e^{rT} - \theta^* = \theta_F\ - \theta^*. \tag{31}$$

Provided the spot price θ_0, or the forward price θ_F, is known, this condition on the statistical regularity $P(\theta)$ is the nonstochastic analogue of the put-call parity condition on the pricing measure of the usual stochastic case. Indeed, in the stochastic case, when $P = \{p\}$ and p is countably-additive, we would have from (31)

$$\int \theta p(d\theta) = \ \theta_0 e^{rT}\ , \tag{32}$$

a condition that locks in the expectation of the stock price and yields the Black-Scholes pricing formula for log-normal $p(\theta)$ with the volatility parameter σ [18].

So, provided the current stock price is θ_0 and the riskless interest rate is r, relations (21), (22), (23), (24) and (31) characterize the forward price of the underlying, the European options prices and the "fair" statistical regularity $P(\theta)$.

2.6. The uncertainty hedge

In the general case of some statistical regularity P, provided the price is the indifference price (11), the minimal expected profits of the uncovered position with pay-off $f(\theta)$ are equal for the buyer and the seller, and are negative. Indeed, substituting (11) back into (9) one has

$$L_Z^*(d_1) = L_Z^*(d_2) = \min_{p \in \boldsymbol{P}} \int L_b(\theta, d_1) p(d\theta) = \ \min_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta) - \ u\, e^{rT} =$$

$$= \ \min_{p \in \boldsymbol{P}} \int f(\theta) p(d\theta) - \ \frac{1}{2}\langle f\rangle_P = -\frac{1}{2}[f]_P\ \ \leq 0. \tag{33}$$

It is easy to see that the maximal expected pay-off of the uncovered option position is in this case positive and equals to $\frac{1}{2}[f]_P\ \ \geq 0$. However, when the statistical regularity P is stochastic, the minimal and the maximal expected pay-offs of the uncovered option position are zero. Relation (33) shows that the use of stochastic probability measures may be the reason why market models and hence risk analysts systematically underestimate the risks of financial transactions.

It is important to note that it is not yet clear whether in the case of statistical regularities of the general form the *dynamic replication* framework is theoretically possible. First of all, we simply do not know yet how to describe the time evolution of statistical regularities. Second, from the valuation view point, the delta hedge construction is not necessary in order to determine the price of the call option: the price is given by (22), (23) and (31). Third, from a risk management perspective, the knowledge of the delta is important, but even having obtained the above pricing relations, the derivation of the option price sensitivity to the change of the underlying price is not trivial.

Nevertheless we can define a *static* hedge. Indeed let us compose a portfolio of a European option and of a position in the underlying and require that its minimal expected pay-off were not negative. Represent this decision as $d^h = (d_1, d_2) \in D \times D = D^2 \in \Delta, d_1 = (f(.), 1, u^*), d_2 = (g(.), \delta, \theta_0)$, where $f(\theta)$ is as above, $g(\theta) = \theta$ in order to represent a position on the underlying itself, u^* is chosen as in (11), θ_0 is chosen as (22) and $\delta \in Q$. Then, according to (6),

$$L(\theta, d^h) = \sum_{i=1}^{2} q_i\left(-u_i e^{rT} + f_i(\theta)\right) = -u^* e^{rT} + f(\theta) + \delta(-\theta_0 e^{rT} + \theta). \tag{34}$$

Then the value of the criterion is

$$L_Z^*(d^h) = \min_{p\in\boldsymbol{P}} \int L \quad (\theta, d^h) p(d\theta) =$$

$$= \min_{p\in\boldsymbol{P}} \left(\left(\int f(\theta)p(d\theta) - u^*\, e^{rT}\right) + \delta\left(-\theta_0 e^{rT} + \int \theta p(d\theta)\right)\right) \geq$$

$$\geq \min_{p\in\boldsymbol{P}} \left(\int f(\theta)p(d\theta) - u^*\, e^{rT}\right) + \delta \min_{p\in\boldsymbol{P}} \left(-\theta_0 e^{rT} + \int \theta p(d\theta)\right) =$$

$$= -u^*\, e^{rT} + \min_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta) + \delta\left(-\theta_0 e^{rT} + \min_{p\in\boldsymbol{P}} \int \theta p(d\theta)\right) =$$

$$= \frac{\min_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta) - \max_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta)}{2} + \delta\, \frac{\min_{p\in\boldsymbol{P}} \int \theta p(d\theta) - \max_{p\in\boldsymbol{P}} \int \theta p(d\theta)}{2}.$$

Requiring the last sum to be equal to zero, one yields:

$$\delta = -\frac{\max_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta) - \min_{p\in\boldsymbol{P}} \int f(\theta)p(d\theta)}{\max_{p\in\boldsymbol{P}} \int \theta p(d\theta) - \min_{p\in\boldsymbol{P}} \int \theta p(d\theta)} = -\frac{[f]_P}{[\theta]_P}. \tag{35}$$

This quantity of the underlying makes the minimal expected pay-off of the transaction $d^h = (d_1, d_2) \in D^1 \times D^1 \in \Delta$ non-negative, as $\min_{p \in \boldsymbol{P}} \int L \ \ (\theta, d^h) p(d\theta) \geq 0$. It is easy to see that the same δ makes the maximal expected pay-off of the transaction $d^h = (d_1, d_2) \in \Delta$ non-positive, i.e. $\max_{p \in \boldsymbol{P}} \int L \ \ (\theta, d^h) p(d\theta) \leq 0$. From these two conditions it follows that

$$\int L \ \ (\theta, d^h) p(d\theta) = 0, \qquad \forall p \in \boldsymbol{P}. \tag{36}$$

This means that the analogue of the delta hedge in this generalized – not necessarily stochastic - setting may guarantee control only over the expectation of the transaction pay-off. This is yet another argument in order to question the dynamic replication framework as it is done in [18].

Though expression (35) could appear at first glance as a "*model risk hedge",* in the light of the broader description of randomness by means of statistical regularities, however, the "model risk" in this sense seems to be a misnomer. Therefore, we prefer to call (35) a *generalized delta* or an *uncertainty hedge*.

3. Discussion

It seems that considering the pricing problem as a problem of choice, or a decision problem, is a natural framework for the static replication argument [18], and, apart from allowing for a consistent introduction of the concept of nonstochastic randomness in mathematical finance, is an effective way to reconcile Arrow-Debreu state preference approach to option pricing and decision theory. This more general framework has enabled us to obtain extensions of classical relations of forward stock price (21), of European options prices (24-25), of conditions on fair statistical regularity (22, 31) as well as an expression for a generalized delta (35). The generalized delta illustrates the idea that in cases of nonstochastic randomness, one can hedge the pay-off of an option position only in expectation. On the other hand, this hedge can be considered as a hedge against what is known as *the model specification risk* However, in the light of the description of randomness by means of statistical regularities, "model risk" seems to be a misnomer. As argued above, the hedge ratio thus defined should rather be regarded as an *uncertainty hedge*.

The pricing formulas are invariant with respect to changes from the uncertainty aversion class Π_1 to the uncertainty proneness class Π_2. Which means that, in the same decision situation $S = (Z, P)$, the same price makes uncertainty averse and uncertainty prone decision makers indifferent between the roles of buyer and of seller of safety. Therefore one is tempted to call this indifference price *uncertainty neutral price*. It seems that for any statistical regularity $P(\theta)$ and any pay-off $f(\theta)$ one can define a unique finitely-additive probability measure p^f out of relation

$$\frac{\min_{p\in\boldsymbol{P}}\int f(\theta)p(d\theta) + \max_{p\in\boldsymbol{P}}\int f(\theta)p(d\theta)}{2} = \int f(\theta)p^f(d\theta). \quad (41)$$

where p^f may be called the *uncertainty neutral finitely-additive probability measure*. This new measure p^f may or may not belong to the original statistical regularity $P(\theta)$.

Among the features that distinguish the treatment of the asset valuation problem proposed in this article, we would like to draw attention to a few items: First, to the treatment of the pricing problem as a decision making issue, or choice problem; second, to the use of linear utility functions (or simply profit and loss functions in our case). These particularities are appealing from a practical viewpoint. First of all, no one knows better than a market maker or a trader that prices he indicates are his decisions. The observed transaction price is the result of his choice, preceded by a negotiation process. This is so for an OTC market, where the negotiation time is long, as well as for any organized market, where the negotiation time is, however, very short. The impact of such a pricing decision on the profit and loss for an important swap transaction could be drastic. All market price movements, especially rallies and sale-offs, are results of combined decisions: buy or sell, at which price, and which quantity. Our framework makes this intuitively clear point precise and seems to offer new possibilities of market microstructure modeling (Appendix B). In particular, we show that market participants' uncertainty profiles are crucial for the bid-ask spreads pricing strategies. It turns out that one can interpret the half value of the difference between the largest and the smallest bid-ask spreads in the limit order book as the *uncertainty price*, the speculative component of the bid-ask spreads. Second, linear utility functions of wealth, or of wealth variations, or simply profit and losses, are consequences of market participants' decisions and constitute what is really of interest to them. This supports the idea to model preferences on consequences by the natural order of real numbers.

At the same time, behavioral particularities of the participants, their aversion, proneness, or neutrality toward uncertainty are described in terms of axioms of preferences on actions (decisions). These preferences on decisions are characterized by convex, concave or linear functionals, requiring existence of the above-mentioned statistical regularities. It is well known that in asset valuation problems one deals, due to the no arbitrage argument, with linear functionals called *pricing operators* [48]. In the context of portfolio choice, requirement of diversification benefits leads to convex functionals similar to (2), called *coherent risk measures* [2]. Traditionally, these two frameworks impose the additional technical requirement that probability measures representing these functional be countably-additive. This requirement drastically misrepresents the reality of randomness, even if it allows using stochastic processes in models which disputably (mis) represent markets.

Decision theory, studying consistent representations of preferences on the set of possible decisions has the potential to combine the two contexts (asset valuation, portfolio selection) without requiring countable-additivity. Decision theory as such supports the idea that all economic decisions, including pricing decisions, are driven by preferences of economic agents. However, the relatively low momentum with which some contemporary versions of decision theory find their way into mathematical finance (comparatively to the expected utility theory [12], [41]), might be linked to their requirements regarding the existence of sets of additive or of non-additive measures, hence in their relation to statistics in general, and to statistical decision theory [15], in particular. The interest in bridging the gap between decision-theoretical set functions and statistics of frequencies has been of interest for quite a while [24], [39], [40], [56]. According to Ivanenko and Labkovsky [29], the closed families of finitely-additive probabilities, which appeared as the consequence of the version of axiomatic decision theory developed in [27] and [28], describe statistically unstable or nonstochastically random mass phenomena. This theoretical result, known as the theorem of existence of statistical regularities, offers an alternative to the traditional point of view, according to which a random phenomenon evolves following a single given probability distribution of which one can state only that it belongs to a certain set of distributions. This traditional interpretation of sets of probability measures is specific to robust statistics [25] as well as to neo-Bayesian decision-theoretical settings, where such sets are called multiple priors [21]. The theorem of existence of statistical regularities is an essential contribution to an alternative view, departing from this traditional interpretation, and provides, in our opinion, the frequentist justification of probability axioms. Paraphrasing A.N. Kolmogorov’s remark, we have no reasons to assume that a real life random phenomenon is characterized by a single probability measure. We consider Mandelbrot’s contributions as going in the same direction.

Last, but not least, it seems reasonable to suggest that further exploration of statistical regularities of nonstochastic randomness may happen to depart at the same time from current underestimations of risk in financial transactions.

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Appendix A. On the definition of uncertainty profile

Uncertainty profile is an important factor in portfolio choice. Indeed, uncertainty averse investors would systematically prefer a neutral or fully diversified investment portfolio to a directional or concentrated one, while uncertainty prone investors would prefer the opposite. The difference between the two types of behavior has its roots in psychology. The uncertainty aversion feature requires from the decision maker, in situations where he has no information about what will happen in the future, to always choose a "middle of the road" solution, if it is available. This principle is known as the guaranteed result principle generalized for mass events [30] and represents one of the forms of uncertainty aversion. To the contrary, the uncertainty proneness feature pushes the decision maker to choose each time solutions as if he knew the path of the future with certainty. In traditional expected utility theory, these effects are modeled by means of the convexity sign of the utility function. In [41] it was shown that the concavity of the utility function is the reason for preference of diversified portfolios. While in the present version of decision theory, these effects are modeled directly by means of axiomatic description of psychological features of a decision maker toward an uncertain future.

We reproduce hereunder the axiomatic definition of the class Π_1 of uncertainty aversion and its characterization theorem [27], [28], [30]. One obtains the characterization of the class Π_2 of uncertainty proneness by means of changing the inequality sign in condition C3 [44], [45].

Definition A1 Let $\mathbb{Z}$ be the class of all ordered triples of the form $Z = (\Theta, U, L)$, where Θ, U are arbitrary nonempty sets and $L: \Theta \times U \to \mathbb{R}$ is a real bounded function. The triple Z is called a decision scheme. We denote by $\mathbb{Z}(\Theta)$ the subclass of all decision schemes of the form $Z = (\Theta, ., .)$, where the set Θ is fixed.

Definition A2 We define a *criterion choice rule* to be any mapping π, defined on $\mathbb{Z}(\Theta)$ and associating to every scheme $Z=(\Theta,U,L)\in\mathbb{Z}(\Theta)$ some real function $L_Z^*(\cdot)$, *a criterion*, determined on U. We denote the class of all criterion choice rules by $\Pi(\Theta)$ and include in the subclass $\Pi_1(\Theta)\subset\Pi(\Theta)$ all criterion choice rules that satisfy the following three conditions:

C1. If $Z_i=(\Theta,U_i,L_i)\in\mathbb{Z}(\Theta), i=1,2, U_1\subset U_2$, and $L_1(\theta,u)=L_2(\theta,u)\ \forall u\in U_1, \forall\theta\in\Theta$, then $L_{Z_1}^*(u)=L_{Z_2}^*(u)\ \ \forall u\in U_1$.

C2. If $Z=(\Theta,U,L)\in\mathbb{Z}(\Theta)$, $u_1,u_2\in U$, then if $L(\theta,u_1)\le L(\theta,u_2), \forall\theta\in\Theta$, then $L_Z^*(u_1)\le L_Z^*(u_2)$, and if $a,b\in\mathbb{R}, a\ge 0$ and $L(\theta,u_1)=aL(\theta,u_2)+b$, $\forall\theta\in\Theta$, then $L_Z^*(u_1)=aL_Z^*(u_2)+b$.

C3. (The principle of guaranteed result) If $Z=(\Theta,U,L)\in\mathbb{Z}(\Theta)$, $u_1,u_2,u_3\in U$ and $L(\theta,u_1)+L(\theta,u_2)=2L(\theta,u_3)\ \forall\theta\in\Theta$, then $2L_Z^*(u_3)\le L_Z^*(u_1)+L_Z^*(u_2)$.

The next Theorem (which is a simplified version of Theorem 1 from [27, 28] or Theorem 2 from [30] or Theorem 5.2 from [31]) establishes the link between the properties of $L_Z^*(\cdot)$ and its structure.

Theorem Criterion $L_Z^*(\cdot)$ possesses the properties C1-C3 if and only if it has the following form

$$(\alpha 1)\qquad L_Z^*(u)=\max_{p\in\boldsymbol{P}}\int L(\theta,u)p(d\theta),$$

where P is a statistical regularity on Θ.

Note that in the above definition and theorem, the sets U,Θ are arbitrary nonempty sets and the loss function $L: U\times\Theta\to\mathbb{R}$ is bounded. If instead of the loss function L one considers profit and loss function $\hat{L}=-L$, as we do in this article, then condition C3 of the above Definition becomes

C3'. If $Z=\left(\Theta,U,\hat{L}\right)\in\mathbb{Z}(\Theta)$, $u_1,u_2,u_3\in U$ and $\hat{L}(\theta,u_1)+\hat{L}(\theta,u_2)=2\hat{L}(\theta,u_3)\ \forall\theta\in\Theta$, then $2L_Z^*(u_3)\ge L_Z^*(u_1)+L_Z^*(u_2)$.

The criterion *(α1)* of maximal expected losses then becomes the criterion of minimal expected utility

$$(\alpha 1')\qquad L_Z^*(u)=\min_{p\in\boldsymbol{P}}\int\hat{L}(\theta,u)p(d\theta).$$

We outline below the demonstration that if a decision maker uses criterion (α1'), then Condition 3 can be interpreted as the diversification argument (see Ivanenko (2010), page 111).

Let as before the decision scheme $Z=(\Theta,D,L(.,.))$ be given by (4-7). Let

$$(\alpha 2)\qquad d_1=(+1;f_1(.);u_1),\quad L(\theta,d_1)=-u_1e^{rT}+f_1(\theta),$$

$(\alpha 3)$ $$d_2 = (+1; f_2(.); u_2), \quad L(\theta, d_2) = -u_2 e^{rT} + f_2(\theta),$$

$(\alpha 4)$ $$d_3 = \left(+1; \frac{f_1(.) + f_2(.)}{2}; \frac{u_1 + u_2}{2}\right),$$

$$L(\theta, d_3) = -\frac{u_1 + u_2}{2} e^{rT} + \frac{f_1(\theta) + f_2(\theta)}{2}.$$

Then it is obvious that

$(\alpha 5)$ $$L(\theta, d_1) + L(\theta, d_2) = 2L(\theta, d_3), \forall \theta \in \Theta.$$

Provided

$(\alpha 6)$ $$L_Z^*(d) = \min_{p \in \boldsymbol{P}} \int L(\theta, d) p(d\theta),$$

show that

$(\alpha 7)$ $$L_Z^*(d_1) + L_Z^*(d_2) \leq 2L_Z^*(d_3).$$

Indeed,

$(\alpha 10)$ $$L_Z^*(d_1) = \min_{p \in \boldsymbol{P}} \int L(\theta, d_1) p(d\theta) = -u_1 e^{rT} + \min_{p \in \boldsymbol{P}} \int f_1(\theta) p(d\theta),$$

$(\alpha 11)$ $$L_Z^*(d_2) = \min_{p \in \boldsymbol{P}} \int L(\theta, d_2) p(d\theta) = -u_2 e^{rT} + \min_{p \in \boldsymbol{P}} \int f_2(\theta) p(d\theta),$$

$(\alpha 12)$ $$L_Z^*(d_3) = \min_{p \in \boldsymbol{P}} \int L(\theta, d_3) p(d\theta) =$$

$$= -\frac{u_1 + u_2}{2} e^{rT} + \min_{p \in \boldsymbol{P}} \int \frac{f_1(\theta) + f_2(\theta)}{2} p(d\theta).$$

Since

$(\alpha 13)$ $$\min_{p \in \boldsymbol{P}} \int f_1(\theta) p(d\theta) + \min_{p \in \boldsymbol{P}} \int f_2(\theta) p(d\theta) \leq \min_{p \in \boldsymbol{P}} \int (f_1(\theta) + f_2(\theta)) p(d\theta).$$

we obtain (α7).

This confirms that to be uncertainty averse means to prefer in situations of uncertainty diversified, or neutral portfolios to directional or concentrated ones. At the same time, uncertainty proneness is modeled by means of a reversion of the inequality, or of the preference sign of actions, in the corresponding conditions C3 and C3’, while keeping the same preference on consequences. This change results in the corresponding change of the extremum type in expression α1 and α1’ (see details in [44], [45]). It can be shown in a similar manner that uncertainty prone investors prefer directional portfolios to diversified ones.

Appendix B. On the nature of bid-ask spreads

The nature of bid-ask spreads in limit order books is a recurrent and important theme in market microstructure research [11], [33], [47]. Traditionally, bid-ask spreads are considered as compensations for providing liquidity. It is assumed in standard financial analysis that bid-ask spreads can result from factors as order processing, adverse information, and inventory holding costs [14], [23]. However, there are still discrepancies between different schools of thought concerning this issue. In particular, it was observed that for a developed limit order book the market participants can be divided at least in two groups, patient and impatient investors, and that patterns of their behaviors and potential market impacts are different [58].

The formalism described in 2.1-2.3 is also a convenient tool to model market microstructure phenomena. We suggest that uncertainty aversion or proneness can be important drivers of bid-ask spread dynamics and factors defining the shape of the limit order book. Uncertainty price can be defined as a speculative component of the bid-ask spread. In the context of the limit order book research, uncertainty averse and uncertainty prone decision makers can be identified, correspondingly, with patient and impatient investors.

Let, as before, decision scheme Z (1), the elements of which are specified by (4-6), represents the one-period Arrow-Debreu model of securities market under uncertainty. Let there be two decision makers, i=1,2, and let them hold their respective views about the random evolution of θ in the form of statistical regularities $P_1(\theta)$ and $P_2(\theta)$ respectively. The decision makers i=1,2 can be buyers, *B*, or sellers, *S*. Decisions of the buyer and the seller are, respectively, $d_B^i = \left(f(\theta), 1, u_B^i\right)$ and $d_S^i = \left(f(\theta), -1, u_S^i\right)$,*i=1,2*. Their actual profits will be, respectively, $L\left(d_B^i, \theta\right) = -u_B^i\, e^{rT} + f(\theta)$ and $L\left(d_S^i, \theta\right) = +u_S^i\, e^{rT} - f(\theta)$. Let there be two rationality classes: uncertainty aversion, Π_1, and uncertainty propensity, Π_2. Decision makers can belong to one of these rationality classes. For each of the rationality classes, the values of the criteria of the buyer and the seller on these decisions are, correspondingly,

$$L_1^*\left(d_B^i\right) = \min_{p\in\boldsymbol{P_i}} \int \left(-u_B^i\, e^{rT} + f(\theta)\right) p(d\theta) = \; -u_B^i\, e^{rT} + \min_{p\in\boldsymbol{P_i}} \int f(\theta) p(d\theta) \qquad \text{(b1)}$$

$$L_1^*\left(d_S^i\right) = \min_{p\in\boldsymbol{P_i}} \int \left(+u_S^i\, e^{rT} - f(\theta)\right) p(d\theta) = \; +u_S^i\, e^{rT} - \max_{p\in\boldsymbol{P_i}} \int f(\theta) p(d\theta). \qquad \text{(b2)}$$

$$L_2^*\left(d_B^i\right) = \max_{p\in\boldsymbol{P_i}} \int (-u_B^i\, e^{rT} + f(\theta)) p(d\theta) = \; -u_B^i\, e^{rT} + \max_{p\in\boldsymbol{P_i}} \int f(\theta) p(d\theta) \qquad \text{(b3)}$$

$$L_2^*\left(d_S^i\right) = \max_{p\in\boldsymbol{P_i}} \int \left(+u_S^i\, e^{rT} - f(\theta)\right) p(d\theta) = \; +u_S^i\, e^{rT} - \min_{p\in\boldsymbol{P_i}} \int f(\theta) p(d\theta). \qquad \text{(b4)}$$

It is reasonable to admit that decision makers would only make those decisions that render positive the values of their respective criteria (minimal or maximal profits). Accordingly, require that (b1)-(b4) be positive. Then, the uncertainty averse decision maker would require their bid and ask prices to satisfy, respectively,

$$u_B^i < e^{-rT} \min_{p \in \boldsymbol{P_i}} \int f(\theta) p(d\theta)\,,\ u_S^i > e^{-rT} \max_{p \in \boldsymbol{P_i}} \int f(\theta) p(d\theta). \tag{b5}$$

while the uncertainty prone decision maker would require their bid and ask prices to satisfy, respectively,

$$u_B^i < e^{-rT} \max_{p \in \boldsymbol{P_i}} \int f(\theta) p(d\theta)\,,\ u_S^i > e^{-rT} \min_{p \in \boldsymbol{P_i}} \int f(\theta) p(d\theta). \tag{b6}$$

Thus the pricing strategy of the uncertainty averse decision maker is more restrictive than the one of the uncertainty prone decision makers. This suggests that bid-ask spreads exist due to uncertainty averse decision makers. Indeed, the indifference price (11) is satisfactory for uncertainty prone decision makers in terms of constraints (b6), but it is not so for uncertainty averse decision makers. In a well developed limit order book, the inside spread comprises a portion of the highest bid-ask spread. Therefore, it is natural to interpret the smallest bid-ask spread as the one devoid of any uncertainty aversion, consisting of only the usually asserted components of this spread (transaction and inventory costs). In sharp contrast to it, the biggest spread should be seen as consisting in all these standard components plus the specific *uncertainty price* that the uncertainty averse decision maker requires the uncertainty prone decision maker to take into account in their transactions. Hence it is reasonable to consider the following relation as the candidate for the uncertainty price

$$\gamma = \frac{1}{2}(S_{max} - S_{min}) = \frac{1}{2}[f]_{P,} \tag{b7}$$

where $[f]_{P,}$ is the statistical variation of the pay-off $f(\theta)$ from Definition 1. One can assume that statistics of such uncertainty prices may reveal properties of the statistical regularities $P(\theta)$.

Using (45)-(46) one can indeed obtain diverse conditions on regularities P_1 and P_2 that make transactions between buyers and sellers impossible. For instance, if the agent *i=1* is the buyer and the agent *i=2* is the seller and they both belong to the uncertainty aversion class, Π_1, then $u_B^1 < e^{-rT} \min_{p \in \boldsymbol{P_1}} \int f(\theta) p(d\theta)$ and $u_S^2 > e^{-rT} \max_{p \in \boldsymbol{P_2}} \int f(\theta) p(d\theta)$. If $\min_{p \in \boldsymbol{P_1}} \int f(\theta) p(d\theta)$ $< \max_{p \in \boldsymbol{P_2}} \int f(\theta) p(d\theta)$, no transaction is possible. One can easily obtain other combinations, but the study of the effects of diverging beliefs is not within the scope of the article.

Thus, according to (b5-b6), the uncertainty averse agent will have the tendency to have high ask prices and low bid prices, while the uncertainty prone agent, to the contrary will have the tendency to lower the ask price and to rise the bid price. The first strategy resembles some “buy cheap sell expensive” strategy, while the second evokes some “buy not so cheap and sell not so expensive” strategy. It is obvious that the first strategy renders transactions more difficult than the second one. This effect leads one to equate uncertainty aversion with "patient" investors and uncertainty proneness with "impatient" ones. This interpretation suggests considering the market as a meeting place not only of buyers and sellers, but of uncertainty averse and uncertainty prone decision makers as well.

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