Utility model with a stationary time discount factor

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Abstract This paper axiomatizes the utility model with an exponential temporal discount rate. An l-PCS with left identity is defined as a PCS with left identity for which the solvability and Archimedean properties are satisfied only related to left-concatenation. This structure has two partial binary operations—multiplication and right division—and a new binary operation is defined on it. Then three conditions are provided to make the l-PCS with left identity into an extensive structure with identity with respect to the newly defined operation. Finally, the utility model is derived from an additive representation on the extensive structure, so that distinct m-period and n-period temporal sequences ($m \neq n$) can be compared.

Keywords Discount factor, Temporal sequence, Weighted additive model, Extensive structure

1 Introduction

The notion of a discounting factor over time might be useful for discussing preferences among temporal sequences of outcomes, such as income streams. Koopmans [7] and Koopmans, Diamond and Williamson [8], assuming the existence of a utility function for temporal sequences over a denumerable future period, studied a set of postulations so as to permit the utility function to represent a preference for advancing the timing of future satisfaction, which is conceptualized as "impatience." In other words, their works formulated such important concepts in terms of utility functions. Furthermore, Krantz *et al.* [9] and Fishburn [4] presented sets of axioms to construct a utility model with discounting factors, restricting temporal sequences to over a finite period. In

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particular, the introduction of "stationarity" by Koopmans [7] reduces the utility model to a special model wherein utilities are discounted at a constant rate. More precisely, let $A^n = A \times \cdots \times A$ be n copies of a nonempty set A and $\succsim$ a preference relation on it. Then the following representation is obtained for all $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in A^n$:

$$(a_1, \ldots, a_n) \succsim (b_1, \ldots, b_n) \Leftrightarrow \sum_{i=1}^{n} \lambda^{i-1} u(a_i) \geqslant \lambda^{i-1} u(a_i),$$

where u is a real-valued function on A and $\lambda \leqslant 1$ is a strictly positive constant. Krantz *et al.* assumed an "additive conjoint structure" and Fishburn considered a finite product of topological spaces to derive this utility model. This gave rise to a problem: comparisons could be made only between temporal sequences over the common period. Further, some of their axioms are difficult to empirically test and seem to be little concerned with decision-making problems for temporal sequences.

Matsushita [11] recently generalized the classical result of Hölder [5] in the context of groupoids (i.e., systems equipped with a binary operation), and developed an axiom system to construct a weighted additive model. From groupoid multiplication, let ab denote the time series of commodities a, b. Then his model is of the following form:

$$u(ab) = \alpha u(a) + u(b), \quad \alpha \geqslant 1.$$

This is actually a utility model with a stationary time discount factor. Indeed, in left-branching fashion, let $(\cdots((a_1a_2)a_3)\cdots)a_n$ denote a temporal sequence of commodities to receive in the next or previous n periods. From the inductive using of the equation of the weighted additive model, we have $u[(\cdots((a_1a_2)a_3)\cdots)a_n] = \sum_{i=1}^{n} \alpha^{n-i} u(a_i)$. Given that u is a ratio scale, dividing both sides by α^{n-1} and setting $\lambda = 1/\alpha (\leqslant 1)$, we obtain

$$u[(\cdots((a_1a_2)a_3)\cdots)a_n]/\alpha^{n-1} = \sum_{i=1}^{n} \lambda^{i-1} u(a_i).$$

Since every temporal sequence can be expressed as a multiplication, distinct m-period and n-period temporal sequences $(m \neq n)$ can be compared.

The aim of this paper is to convert the axioms into a general version. In short, the axioms are to be rewritten under the requirement that the multiplication be generalized to a partial binary operation; in addition, an interpretation is put on several axioms in the context of decision-making problems of temporal sequences. The rest of this paper is organized as follows. Section 2 provides the axioms to define a basic structure, called l-PCS with left identity, and shows some properties that are satisfied on the structure. Section 3 presents three conditions to make every l-PCS with left identity an extensive structure with identity related to an introduced operation, and gives the main theorem for the weighted additive model. Section 4 contains several conclusions. The proofs of the propositions and theorem along with the lemmas are given in Section 5.

2 Basic concepts

Throughout this paper, $\mathbb{R}^+$ denotes the set of all non-negative real numbers. Further, let $\succsim$ be a binary relation on a nonempty set A that is interpreted as a preference relation. As usual, $\succ$ denotes the asymmetric part, $\sim$ the symmetric part, and $\precsim$, $\prec$ denote reversed relations. The binary relation $\succsim$ on A is a *weak order* if and only if it is connected and transitive. Let $\cdot$ be a "partial" binary operation on A. The operation means a function from a subset B of $A \times A$ into A. The expression $a \cdot b$ is said to be *defined* if and only if $(a,\ b) \in B$. An element $e \in A$ denotes no change in the status quo with temporal sequences. That is, it is considered that receiving e prior to a is no different from receiving a at present; however, ae implies advancing the receipt of a by one period, so that ae is not always $\sim a$.

Henceforth, in the following conditions, the indicated operations $\cdot$ are always assumed to be defined.

A1. Weak order: $\succsim$ is a weak order on C.
A2. Local definability: if $a \cdot b$ is defined, $a \succsim c$, and $b \succsim d$, then $c \cdot d$ is defined.
A3. Monotonicity: $a \succsim b \Leftrightarrow a \cdot x \succsim b \cdot x \Leftrightarrow x \cdot a \succsim x \cdot b$ for all $a,\ b,\ x \in A$.
A4. Left identity: e is a *left identity element*, i.e., $e \cdot a \sim a$ for all $a \in A$.

The system $\langle A,\ \succsim,\ \cdot \rangle$ is a *concatenation structure* if and only if A1–A3 are satisfied. If, in addition, A4 holds, then $\langle A,\ \succsim,\ \cdot,\ e \rangle$ is said to be a *concatenation structure with left identity*. Further fundamental conditions for concatenation structures are listed below.

A5. R-positivity: whenever $a \in A$, then $x \cdot a \succsim x$ for all $x \in A$.
A6. Restricted left solvability: $a \succ b$ implies that $x \cdot b \sim a$ for some $x \in A$.

Axiom A5 is defined as the "right sided" concept, whereas A6 is defined as the "left sided" concept. That is, r-positivity is the positivity condition that is satisfied only for the right-concatenation by a. Restricted left solvability is a generalized restricted solvability in the sense that only the existence of a left solution is permissible. Since x is uniquely determined up to $\sim$ by A3, we write $x = a/b$. This defines a partial binary operation $/$ on A, which is called a *right division*. Indeed, $/$ is a function from the subset $\{(a,\ b) \in A \times A\,|a \succsim b,\ (x,\ b) \in B$ for some $x \in A\}$ into A. It may be rational to refer to A6 as *right divisibility*. If a concatenation structure contains a left identity element e, then by A3 a is l-positive (i.e., $a \cdot x \succsim x$ for all $x \in A$) if and only if $a \succsim e$, whereas a strictly positive element $a \succ e$ is not always r-positive (see Example 1 below). However, the following is satisfied.

Proposition 1 *Let $\langle A,\ \succsim,\ \cdot,\ e \rangle$ be a concatenation structure with left identity. If A is r-positive, then $a \succsim e$ for all $a \in A$.*

Proposition 2 *Let $\langle A,\ \succsim,\ \cdot,\ e \rangle$ be a concatenation structure with left identity. If A6 holds, then for all $a,\ b,\ x \in A$,*

(i) $(a \cdot b)/b \sim a \sim (a/b) \cdot b$.
(ii) Monotonicity of right division [3, Lemma 3.1]: $a \succsim b \Leftrightarrow a/x \succsim b/x,\ x/a \precsim x/b$.

Example 1 We define a binary operation $\oplus$ on the set $\mathbb{R}^+$ by

$$a \oplus b = \alpha a + b \quad \text{for some } \alpha > 0.$$

The set $\mathbb{R}^+$ with this operation and the usual order $\geqslant$ is a concatenation structure with a left identity element 0. Since

$$\alpha < 1 \Rightarrow a \oplus 0 < a \quad \text{for all } a \in \mathbb{R}^+,$$
$$\alpha > 1 \Rightarrow a \oplus 0 > a \quad \text{for all } a \in \mathbb{R}^+,$$

it turns out that $\mathbb{R}^+$ is r-positive for $\alpha > 1$, but not for $\alpha < 1$. Moreover, in both cases of $\alpha > 1$ and < 1, it is seen that whenever $a > b$, then $x = (a - b)/\alpha$ is a left solution to $x \oplus b = a$, but a right solution $x \geqslant 0$ does not always exist to $b \oplus x = a$.

A relaxed version [6] of the Archimedean property is provided. In this regard, we will inductively define the n-th "left" multiple of an element a by $a^0 = e,\ a^1 = a$ and

$$a^n = a \cdot a^{n-1} \text{ if the right-hand is defined}$$
$$a^n \text{ is undefined otherwise.}$$

A concatenation structure is said to be *left Archimedean* if every bounded sequence $\{a^n\}$ constructed as above is finite:

A7. Left Archimedean: every bounded sequence $\{a^n\}$ consisting of the left multiples of a is finite.

It is worthwhile to recall a generalized concept of an extensive structure: a *PCS* [10, Definition 19.3] is a concatenation structure $\langle A, \succsim, \cdot\rangle$ that is positive (i.e., $x \cdot a \succsim x, a$ for all $(x, a) \in B$), and for which A6[1] and A7 are satisfied for both left- and right-concatenations.

Definition 1 An *l-PCS with left identity* is a concatenation structure $\langle A, \succsim, \cdot, e\rangle$ with left identity for which Axioms A5, A6, and A7 are satisfied.

Remark 1 From Proposition 1 it can be seen that an l-PCS with left identity consists at most of positive elements; as such, by the statement immediately before the proposition it is also l-positive. That is, an l-PCS with left identity is a PCS with left identity for which the restricted solvability and Archimedean properties are satisfied only related to left-concatenation.

We now introduce the following conditions given that the indicated operations $\cdot$ are defined:

- Weak associativity: $(a \cdot b) \cdot c \sim a \cdot (b \cdot c)$.
- Weak commutativity: $a \cdot b \sim b \cdot a$.

From Definition 19.3 [10], a weakly associative PCS is an *extensive structure* (see [9] for the formal definition). Hence we see that every l-PCS with left identity is an extensive structure with identity as long as weak associativity and weak commutativity hold. Here weak commutativity is essential, because it can turn Axioms A5–A7 into right and left sided concepts.

Remark 2 [9, Theorem 3.3] If $\langle A, \succsim, \cdot\rangle$ is an extensive structure, then there exists a real-valued function u on it having the following properties:

- $a \succsim b \Leftrightarrow u(a) \geq u(b)$;
- $u(a \cdot b) = u(a) + u(b)$ whenever $a \cdot b$ is defined.

Moreover, this representation is unique up to multiplication by a positive constant.

An *additive representation* on A is a real-valued function satisfying the order-preserving and additivity properties.

[1] Restricted solvability is as follows: $a \succ b$ implies that $a \succsim x \cdot b \succ b$ for some $x \in A$.

3 Time discounting utility

Henceforth, assume that an l-PCS $\langle A, \succsim, \cdot, e\rangle$ with left identity has no smallest strictly positive element. Concatenations expressed implicitly by juxtaposition are meant to bind more strongly than the right divisions so as to reduce the number of brackets in equalities. For example, $(a \cdot b)/b$ reduces to ab/b. For any $a \in A$, we denote the mappings of a subset of A into A defined by the rules $R_a(x) = xa$ and $L_a(x) = ax$ by R_a and L_a, respectively.

Define a partial binary operation $\circ$ on A by

$$a \circ b = (a/e)b. \tag{1}$$

We make a comment on the domain of $\circ$. Assume that $a \in A$. Then by Proposition 1, $a \succsim e$, so by A6, a/e exists in A. Since e is to belong to A, by A5, $ae \succsim a$, so by Proposition 2, $a \succsim a/e$. Hence we obtain from A2 that if $(a, b) \in B$, then $(a/e, b) \in B$; this implies that B is at least contained in the domain of $\circ$.

The following conditions[2] are needed to construct our time discounting utility. Whenever the indicated operations $\cdot$ are defined, then for all $a, b, c \in A$,

$$((a/e)b/e)c \sim (a/e)((b/e)c); \tag{2}$$

$$(a/e)b \sim (b/e)a; \tag{3}$$

$$((a/e)b)e \sim (ae/e)(be). \tag{4}$$

Identities (2) and (3) specify the weak associativity and commutativity of $\circ$, respectively, i.e., $(a \circ b) \circ c \sim a \circ (b \circ c)$ and $a \circ b \sim b \circ a$. Identity (4) implies that the right multiplication by e is a homomorphism of a subset of A into A related to $\circ$, i.e., $R_e(a \circ b) \sim R_e(a) \circ R_e(b)$.

It will be useful to place an interpretation on these identities in the context of decision-making problems. Let $(ab)c$ or $a(bc)$ denote the outcome of receiving a, b, and c in periods 1, 2, and 3, respectively. Given that $ae/e \sim a$, it will be rational to interpret a/e to mean that the receipt of a is postponed by one period. Then both sides of (2) show that a person receives a, b, and c in period 3. Indeed, the left-hand side shows that first the receipt of a shifts to period 2, and then the receipt of a and b to period 3; the right-hand side means that the person first receives b and c in period 3, and then receives a in period 3, which is postponed by two periods, noting that $a/e \sim (ae/e)/e$. Similarly, both sides of (3) imply that the receipt of a and b is in period 2. Hence (2) and (3) assert that as long as commodities are received in the same period, the concatenation operation is weakly associative and commutative. Further, (4) gives that advancing the receipt of a and b in period 2 by one period is equivalent to receiving a and b in period 1. This interpretation will be suitable to decision-making problems of this kind.

By (3), we may define the n-th addition of a in the left sided manner: $na = a \circ (n-1)a$ if the right-hand side is defined for $n = 2, 3, \ldots$ and $1 \cdot a = a$. From (1) it is seen that $na = L_{a/e}^{n-1}(a)$ for $n \geqslant 1$ where $L_{a/e}^{0} = L_e$.

Lemma 1 *Let A be an l-PCS with left identity. If (2) and (3) are satisfied, then $B(A) = \langle A, \succsim, \circ, e\rangle$ is an extensive structure with identity.*

[2] According to [11], these are properties for "central" r-naturally fully ordered groupoids with left identity to satisfy.

Theorem 1 *Let A be an l-PCS with left identity for which (2) to (4) are satisfied. Then there exist a real number $\alpha \geqslant 1$ and a real-valued function u on A such that*

(i) $a \succsim b \Leftrightarrow u(a) \geqslant u(b)$;
(ii) $u(ab) = \alpha u(a) + u(b)$ *whenever* $(a, b) \in B$;
(iii) $u(e) = 0$.

Moreover, another real number $\alpha' \geqslant 1$ and function u' satisfy (i), (ii), and (iii) if and only if $\alpha' = \alpha$ and $u' = \gamma u$ for some real number $\gamma > 0$.

Remark 3 Let $(a_1a_2)a_3$ and $b_1b_2 \in A$ be three-period and two-period temporal sequences, respectively. For example, assume that $(a_1a_2)a_3 \succsim b_1b_2$. From (i), (ii) of the theorem we obtain

$$(a_1a_2)a_3 \succsim b_1b_2 \Leftrightarrow \alpha^2 u(a_1) + \alpha u(a_1) + u(a_1) \geqslant \alpha u(b_1) + u(b_2).$$

The hypothesis of the following corollaries is that A is an l-PCS with left identity.

Corollary 1 *If $A = \mathbb{R}^+$ and if $\succsim$ and $\circ$ equal the usual order $\geqslant$ and addition $+$, respectively, then $ab = \alpha a + b$ $(\alpha \geqslant 1)$ for all $a, b \in \mathbb{R}^+$.*

Corollary 2 *If e is a two-sided identity, then A is an extensive structure with identity.*

4 Conclusion

This paper axiomatized the utility model with an exponential temporal discount rate. The concept of an l-PCS with left identity was introduced. This structure has two partial binary operations, multiplication and right division, and its left identity e has an important meaning: division of a commodity by e implies postponing its receipt by one period. Using the division by e, a new binary operation was defined. Then three conditions (which seem to be rational in the context of decision-making problems for temporal sequences) were provided so as to make the l-PCS with left identity an extensive structure with identity with respect to the newly defined operation. Finally, the utility model was derived from an additive representation on the extensive structure. This enables us to compare m-period and n-period temporal sequences where $m \neq n$. A topic for future research is the axiomatization of a "generalized" weighted additive model, $u(ab) = \alpha(b)u(a) + u(b)$.

4.1 Proposition 1

Proof. By A5 we have $aa \succsim a$ for all $a \in A$, i.e., $aa \succsim ea$ by A1 and A4. Hence by A3 we obtain $a \succsim e$ for all $a \in A$. □

4.2 Proposition 2

Proof. (i) By definition, $(a/b)b \sim a$. Note that by A3, $x \sim a$ is a unique solution to $xb \sim ab$. Hence we obtain $ab/b \sim a$.

(ii) Since $a \sim (a/x)x$ and $b \sim (b/x)x$ by (i), we have by A3 $a \succsim b \Leftrightarrow a/x \succsim b/x$. Further, repeated use of A3 gives $a \succsim b \Leftrightarrow (x/a)a \succsim (x/a)b \Leftrightarrow x \succsim (x/a)b \Leftrightarrow (x/b)b \succsim (x/a)b \Leftrightarrow x/b \succsim x/a$. □

4.3 Lemma 1

Proof. Axiom A1 is obvious for $B(A)$. As was stated immediately after (4), $B(A)$ is weakly associative and commutative. Hence from the statement before Remark 2 we may prove that A2–A7 hold for $B(A)$. Recall here that by virtue of commutativity, the validity of each axiom may be shown in either the left or right sided manner.

A2 for $B(A)$. Recall that if $a \succsim e$, then $a/e \in A$ (see the statement after (1)). By (ii) of Proposition 2, we have $a \succsim c \Leftrightarrow a/e \succsim c/e$. If $b \succsim d$, then by A2, $(a/e,\, b) \in B \Rightarrow (c/e,\, d) \in B$.

A3 for $B(A)$. By A3 and (ii) of Proposition 2, we have

$$a \succsim b \Leftrightarrow (a/e)c \succsim (b/e)c \text{ and } a \succsim b \Leftrightarrow (c/e)a \succsim (c/e)b.$$

A4 for $B(A)$. Since $e/e \sim e$, it follows from A3 and A4 that $e \circ a \sim a$.

A5 for $B(A)$. If $x \succsim e$, then by (ii) of Proposition 2, $x/e \succsim e$. By A3 we have $(x/e)a \succsim (x/e)e$, or $(x/e)a \succsim x$.

A6 for $B(A)$. By A6, let $x \in A$ be such that $a \sim xb$ whenever $a \succsim b$. Since $(x,\, b) \in B$ and $b \succsim e$, A2 guarantees that $(x,\, e) \in B$. Hence we can set $s = xe$ to obtain $a \sim (s/e)b$.

A7 for $B(A)$. Assume to the contrary of the Archimedean property that there exists a bounded infinite sequence $\{na\}$, $a \in A$. Since $ae \succsim a$ by A5, we have by Proposition 2 $a \succsim a/e$. Since the map $L_{a/e}^{n-1}$ is order preserving, it follows that $na = L_{a/e}^{n-1}(a) \succsim L_{a/e}^{n-1}(a/e)$. This implies the existence of a bounded infinite sequence $\{(a/e)^n\}$, which contradicts the left Archimedean property for A. □

4.4 Theorem 1

We provide a concept for weakly ordered sets. Let B and C be nonempty sets that are weakly ordered with respect to the same relation $\succsim$. Then B and C are *equivalent* to each other if for every $y \in B$ there exists a $z \in C$ such that $y \sim z$, and vice versa.

Proof. Since $B(A)$ is an extensive structure with identity by Lemma 1, it is seen from Remark 2 that there exists an additive representation u on $B(A)$. Then since $u(e) = u(a \circ e) = u(a) + u(e)$, we obtain $u(e) = 0$. In view of A2, the hypothesis $(a,\, b) \in B$ implies that $(a,\, e) \in B$. Since $ab = ae \circ b$, $u(ab) = u(ae) + u(b)$. To complete the proof, it suffices to show that $u(ae) = \alpha u(a)$ for some $\alpha \geqslant 1$. For this the following lemma is provided.

Lemma 2 *Let $A_e = \{ae \,|\, a \in A,\ (a,\, e) \in B\}$. Then A_e and A are equivalent to each other, and hence $B(A_e) = \langle A_e, \succsim, \circ, e\rangle$ is an extensive structure with identity.*

Proof. Since it is obvious that $ae \in A_e$ is contained in A, we show only that for every $x \in A$ there exists an $ae \in A_e$ such that $x \sim ae$. Let $x \succsim e$ be an arbitrary element of A. Then since $x/e \in A$, $a = x/e$ is a solution to $x \sim ae$. It is clear from Lemma 2 that $B(A_e)$ is an extensive structure with identity. □

Combining this lemma with Remark 2, we can obtain that u is also an order-preserving additive function on $B(A_e)$. In view of (4),

$$u((a \circ b)e) = u(ae \circ be) = u(ae) + u(be).$$

Define $u_e(a) = u(ae)$ for all $a \in A$ with $(a, e) \in B$. Then the equation shows that u_e is an additive representation on $B(A)$. Hence by the uniqueness assertion of Remark 2 there is a strictly positive real number α such that $u_e(a) = \alpha u(a)$. Moreover, since $a \precsim ae$ for all $a \in A$ by A5, $u(a) \leqslant u(ae) = \alpha u(a)$. Thus $\alpha \geqslant 1$. Finally, we prove the uniqueness assertion. Assume that α' and u' satisfy (i), (ii), and (iii). Then since u' is an additive representation on $B(A)$, by the uniqueness assertion we have $u' = \gamma u$ for some $\gamma > 0$. However, since $u_e(a) = u(ae)$ and $u'_e(a) = u'(ae)$, $u'_e = \gamma u_e$ must be valid, and hence $\alpha' u' = \gamma\alpha u$. Eliminating u from the equations $u' = \gamma u$, $\alpha' u' = \gamma\alpha u$ and noting that the resulting equation holds for all $a \in A$ with $(a, e) \in B$, we have $\alpha' - \alpha = 0$, or $\alpha' = \alpha$, as required. □

4.5 Corollary 1

Proof. It suffices to show that the function u in the proof of Theorem 1 is continuous. Indeed, if so, then since u is additive and continuous on $\mathbb{R}^+$, it is well known [1] that $u(a) = sa$ for some $s \in \mathbb{R}$. Setting $s = 1$, we obtain $ab = \alpha a + b$. To prove continuity, assume that $a > b$. By A6 $a = xb$ for some $x \in A$. Since A has no smallest strictly positive element, we have $a > x'b > b$ for $x' < x$, and hence $u(a) > u(x'b) > u(b)$. This implies that u has no gap in its range. Hence we conclude from Debreu's [2] open gap lemma that u is continuous. □

4.6 Corollary 2

Proof. Since $a/e \sim a$, identities (2) and (3) reduce to $(ab)c \sim a(bc)$ and $ab \sim ba$, respectively. By the statement immediately before Remark 2, A is an extensive structure with identity. □

References

1. Aczél, J., *Lectures on functional equations and their applications*, Academic Press, New York (1966).
2. Debreu, G., Continuity properties of Paretian utility, *International Economic Review*, **5** 285 − 293 (1964).
3. Demko, M., Lexicographic product decompositions of partially ordered quasigroups, *Math. Slovaca*, **51**, 13 − 24 (2001).
4. Fishburn, P. C., *Utility theory for decision making*, Wiley, New York (1970).
5. Hölder, O., Die axiome der quantität und die lehre vom mass, *Ber. Verh. Sächs. Ges. Wiss. Leipzig Math.-Phys. Cl.*, **53**, 1 − 64 (1901).
6. Iseki, K., Structure of special ordered loops, Portugal. Math., **10**, 81 − 83 (1951).
7. Koopmans, T. C., Stationary ordinal utility and impatience, *Econometrica*, **28**, 287 − 309 (1960).
8. Koopmans, T. C., Diamond, P. A., and Williamson, R. E., Stationary utility and time perspective, *Econometrica*, **32**, 82 − 100 (1964).
9. Krantz, D. H., Luce, R. D., Suppes, P., and Tversky, A., *Foundations of measurement* (Vol. 1), Academic Press, New York (1971).
10. Krantz, D. H., Luce, R. D., Suppes, P., and Tversky, A., *Foundations of measurement* (Vol. 3), Academic Press, New York (1990).
11. Matsushita, Y., Central r-naturally fully ordered groupoids with left identity, *Quasigroups and Related Systems*, **19**, 287 − 300 (2011).