

# Voting cycles when a dominant point exists

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## Abstract

We consider a spatial model of electoral competition between two political parties. These parties are incompletely informed about voter preferences and search adaptively, by means of polling, for those policy platforms which might make them win the election. We introduce interest groups in this spatial framework. Different interest groups coordinate voting behavior and transmit information about voter preferences to the political candidates. Moreover, organization of voters into different interest groups occurs endogenously. We investigate the dynamics of this spatial model of electoral competition by looking at the mean-dynamics, i.e. by replacing stochastic variables by their expected values. The resulting Markov process shows that voting cycles exist. The mechanism driving these voting cycles may explain some empirical regularities found in the political science literature.

*Keywords:* Spatial voting models, electoral competition, interest groups, voting cycles.

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## 1 Introduction

Existing models of electoral competition typically make strong assumptions about the information political parties and voters have regarding issues that are of importance to their decision making. Take the classical Downs-Hotelling spatial competition model. In this model, the preferred policy of a voter is modeled as a point in some kind of policy or issue space. Voters are taken to vote for the party whose policy platform is closest (e.g. in terms of Euclidean distance) to this ideal point. Each voter is assumed to be able to evaluate the consequences of all policy positions and to

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have a stable (complete and transitive) preference ordering over all these positions. In its turn, a political party is assumed to have complete information about the distribution of the ideal points of the voters when selecting the platform that optimizes its electoral prospects. These assumptions are extremely demanding and, in fact, not realistic. Why would parties and candidates spent so much money and effort on election campaigns, for example, if this were not true? And how precisely do parties actually get to know the political consequences of alternative policy platforms? With respect to voters, social interaction may influence their political preferences, as is further manifested by hypes and herding behavior.

Although we certainly do not want to argue against the use of simplifying assumptions in exploring political economic issues of interest, we think it is important also to examine models that take the strong informational constraints in politics into account. In this paper we will do so using a spatial competition model with two office motivated parties. Starting point is the observation that parties do not just know what makes voters tick but have to find out through some kind of polling. Our model therefore allows for the transmission of information on political preferences via polling. However, this search activity is costly. For two reasons, voters may be willing to contribute in the form of effort or money. Firstly, because in this way they can affect the election outcome (electoral motive). Secondly, by conditioning their contributions they affect policies (influence motive). Conditioning takes place by making contributions only available for polling in that part of the political issue space that the voter is mostly concerned about. This setup seems quite realistic. For example, in the case of U.S. presidential elections voters contribute to (new) candidates within parties who are willing to represent their concerns (see e.g. the website [deanforamerica.com](http://deanforamerica.com)), which effectively implies that the amount of polling on policy stances related to these concerns is increased. If the candidate is successful (s)he will win the race and can implement the policies preferred by the voters who supported this candidate. Of course, instead of contributing directly to candidates or parties, voters may choose an indirect way by supporting intermediary institutions with the same purpose (see e.g. the website [meetup.com](http://meetup.com)). For simplicity, we will not distinguish between these different channels, but have voters contribute to an 'interest group' which conditionally transfers the contributions to the parties. In line with the evidence presented in Ansolobehere, de Figueiredo and Snyder (2002), contributions are not modeled as resulting from an explicit investment calculus, but assumed to be primarily driven by dissatisfaction with existing policies on issues of particular concern to the voter. Note that by getting politically involved in this way voters are likely to identify themselves with the policy stances and institutions they go for. In our model it is assumed, therefore, that some coordination of voting will occur, because of the extra weight that voters will attach to these positions on issues in their votes. Clearly, this coordination of voting may affect policies.

By having political parties experiment before an election with different policy positions to improve their chances of winning, our study is related to Kollman, Miller and Page (1992, 1998). An important goal of their research was to investigate the relevance of the theoretical "chaos" results for multi-dimensional issue spaces, which predict that, in general, the incumbent would always be defeated by the challenging

party (see e.g. McKelvey, 1976, 1979, and Schofield, 1978). Instead of “chaos”, however, the simulations of Kollman, Miller and Page (1992) showed convergence of the parties’ platforms to the center of the distribution of voters’ ideal positions. Sadiraj, Tuinstra and van Winden (2004) presents an extensive simulation study of a spatial competition model with endogenous emergence of interest groups. This simulation study suggests that the presence of interest groups increases the probability of winning for the challenger and also increases separation between policy platforms. Moreover, the policy outcome seems to: (a) move in the direction of the center of the distribution of voter preferences, though at a lower speed; and (b) stabilize at a distance 0.6 from the center. In this paper we elaborate on these results some more. We introduce the *mean dynamics*, where we replace the stochastic elements of the model by their expected values. This results in a Markov model, for which we can investigate the steady state distribution and the stability properties. We are interested in: *i*) the distance between the policy outcome and the center of the space, and *ii*) the probability that the challenger wins an election, both of which can be derived once the transition and initial probabilities are known. In particular, the asymptotic properties of the models are of interest. It turns out that the steady state distribution of policy outcomes depends critically upon the way interest groups transmit information about the electoral landscape to the political parties. We will rely on simulations to investigate the relevance of the results for stochastic versions of the model. Finally, we will show that the model of interest groups may help explain some “stylized facts” concerning empirical data on policy outcomes.

The rest of the paper is organized as follows. In Section 2 the adaptive political party model is discussed. This model is taken from Sadiraj, Tuinstra and van Winden (2004). In Section ?? the mean-field approximations of the stochastic models are introduced. In Section 3 the dynamics of the specified models are analyzed. Section 4 presents some results that shed light on the emergence of voting cycles in the presence of the interest groups. Section 5 is concerned with the relevance of the results for the stochastic models and replication of some stylized facts. Section 6 concludes.

## 2 The model

### 2.1 Incompletely informed political parties in spatial competition

We consider a typical and standard spatial model of electoral competition (for an introduction into the spatial theory of voting, see Enelow and Hinich, 1984). Policy platforms are represented as points in an issue space. To keep the model tractable we restrict our attention to a two-dimensional issue space, with  $K$  positions per issue. Hence, the issue space is  $\mathcal{X} = \{1, \dots, K\} \times \{1, \dots, K\}$ . There is a population of  $N$  voters, which evaluate different policy platforms according to the weighted distance to their own preferred policy platform. That is, given the voter’s *ideal point*  $x_j \in \mathcal{X}$ ,

the utility a voter attaches to a policy outcome  $y = (y_1, y_2)$  corresponds to

$$u_j(y) = - \sum_{i=1}^2 s_{ji} (x_{ji} - y_i)^2, \quad (1)$$

where  $s_{ji}$  corresponds to the weight or *strength* voter  $j$  attaches to issue  $i$ . With respect to these strengths, we assume that  $s_j = (s_{j1}, s_{j2}) \in \mathcal{S} \times \mathcal{S} = \{s_0, s_1, \dots, s_c\} \times \{s_0, s_1, \dots, s_c\}$ , with  $0 \leq s_0 < s_1 < \dots < s_c \leq 1$ . In this model voter preferences are completely identified by  $(x_j, s_j) \in \mathcal{X} \times \mathcal{S}^2$ . A particular configuration of voters is generated by drawing, for each voter  $j$ , an ideal position  $x_j$  from the discrete uniform distribution on  $\mathcal{X}$  (hence there is no correlation between the positions on the different issues) and strengths  $s_{ji}$  from a discrete distribution on  $\mathcal{S}$  (hence strengths are uncorrelated).

Given the initial configuration of voters an *electoral landscape* can be constructed as follows. There are two political parties entering the election, the incumbent and the challenger. It is assumed that the incumbent does not change its policy from the previous period. Each voter votes for the political candidate yielding him the highest utility as given by (1). Then for each position the height of the electoral landscape is determined as the fraction of voters voting for the challenger, if it would select that position. That is, we can define  $h(z | y)$  as the fraction of the votes the challenger gets if the incumbent is at platform  $y$  and if the challenger selects platform  $z$ . For every policy position  $z$  with  $h(z | y) > (<) \frac{1}{2}$ , the challenger wins (loses) the election. For a policy position with  $h(z | y) = \frac{1}{2}$ , the challenger wins with probability  $\frac{1}{2}$ . The objective for the challenger is to find high points of the electoral landscape. This is consistent with a situation where candidates are only interested in getting elected and therefore search for the highest position (which will make them harder to beat in the next election if they win this election). The problem of the challenger therefore reduces to a search problem: it has to find the (local) optimum of some complicated nonlinear function (i.e. the electoral landscape  $h(\cdot | y)$ ). Instead of assuming that political parties or candidates have complete information about the electoral landscape, which seems rather farfetched anyway, we follow Kollman, Miller and Page (1992) in assuming that political parties have incomplete information about voter preferences and select policy platforms *adaptively*. We consider the following adaptive search procedure. The challenger randomly draws a number of positions from the issue space and runs a poll there. Such a poll consists of, for example, a randomly drawn 10% of the voters. The challenger observes the fraction of this poll which favors his policy over the incumbents policy and uses this as an estimate of the true height of the electoral landscape at that position. If the best polling result indicates a height of at least  $\frac{1}{2}$  then the challenger chooses that position. Otherwise it chooses the incumbent position, where it has probability  $\frac{1}{2}$  of winning the election. If the true height of the landscape at the position selected by the challenger is above (below)  $\frac{1}{2}$ , the challenger (incumbent) wins the election. If the height is exactly  $\frac{1}{2}$ , each political party has a probability  $\frac{1}{2}$  of winning the election. This procedure is repeated for each election that follows.

## 2.2 Interest groups

The observation that political parties are incompetently informed about voter preferences introduces an incentive for special interest groups to emerge. In this paper we will, rather than just assuming that these interest groups exist and their preferences and size (i.e. influence) are given exogenously, model interest groups as endogenously emerging institutions, arising from social pressure and herd behaviour of individual agents. Interest groups have several types of functions. Firstly, through these interest groups the voting behaviour of the members is coordinated. Secondly, they provide information to the political candidates in order to influence the outcome of the election process. Our approach therefore differs from most of the literature on interest groups which focuses on lobbying and campaign contributions and uses game theoretic models to describe the interaction between political parties and interest groups (for surveys of this literature we refer to Austen-Smith (1994) and van Winden (1999)).

Interest groups emerge in our model of spatial competition as follows. Voters with the same ideal position on one of the issues, say the first issue, may decide to organize in interest groups in order to play a, hopefully pivotal, role in determining the election outcome. Notice that interest groups are only concerned with one issue. Now let  $n_k^i$  be the total number of voters having position  $k \in \{1, \dots, K\}$  on issue  $i \in \{1, 2\}$ . Clearly, we have  $\sum_{k=1}^K n_k^1 = \sum_{k=1}^K n_k^2 = N$ . Now along each of the  $2K$  “lines” in the issue space an interest group emerges. Prior to each election a process of interest group formation takes place. In this process each voter determines whether he or she will join one or two interest groups. After this process of interest group formation is over, it is endogenously determined which interest groups become “active”. In particular, we assume that this depends upon whether the interest group has collected enough funds to financially support one of the candidates.

Each voter is a potential member of two interest groups and has to decide whether to join none, one or both interest groups. This decision is modelled as follows. The incentive to join the interest group is that this provides a means to exerting some political influence. Clearly, the larger the interest group the higher the probability of having some influence, so this incentive is increasing in the number of interest group members. There may also be some positive feedback because one can identify oneself with this interest group. Furthermore, we assume that voters are more inclined to join an interest group if the present policy position on that issue is farther away from their own position on that issue. On the other hand, there are some costs of joining the interest group. In this paper we consider the cost of joining as exogenous and fixed<sup>1</sup>. The process is modeled as follows. Potential interest group members are drawn in a random order and sequentially determine whether to join or not. This procedure is repeated once (only for the voters who have not joined), so each voter has to decide whether to join or not one or two times. Let  $m_{k,s-1}^i$  be the number of members of the interest group at position  $k$  of issue  $i$  after  $s - 1$  voters have been

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<sup>1</sup>We can extend this model by incorporating other types of costs as well. For example, joining an interest group implies following the interest group’s advice about how to cast the vote in the upcoming election, although this might be bad in terms of the election outcome for the other issue. The costs for somebody joining the interest group therefore depend upon his strength and ideal position in the other issue (or alternatively, on the relative strength of the relevant issue).

drawn and have made the decision whether to join a certain interest group or not (where  $m_{k,0} \equiv 0$ ). The  $s$ 'th voter then decides on the basis of the following general decision rule

$$v_{js}(k) = V \left( k, \frac{m_{k,s-1}^i}{n_k^i}, y \right) \quad (2)$$

which we specify as

$$v_{js}(k) = s_{ji} (k - y_i)^2 \exp \left( 1 + \frac{m_{k,s-1}^i}{n_k^i} \right) - c \quad (3)$$

where  $c$  corresponds to exogenous costs or benefits that are incurred by joining an interest group. Notice that the incentive to join the interest group increases with the strength  $s_{ji}$ , increases with the distance between the incumbent's platform  $y_i$  and the voter's ideal position on the issue ( $k$ ) and increases with the size of the interest group. Each potential interest groups undergoes a similar formation process, leading to  $2K$  different interest groups, each of which has a certain size.

Now that we have modeled the emergence of interest groups, let us consider their functioning. In our model interest groups influence the election process in three ways: they coordinate the voting behavior of their members, they provide information about the electoral landscape to the political candidates, and they try to influence policy outcome by imposing conditions on polling. Let us first discuss what information they provide. Each interest group possesses certain funds raised by contribution fees  $c$  of its members. These funds are offered to the challenger conditional on: *i*) running a certain number of polls in policy positions coinciding with the interest group's position on the relevant issues; *ii*) commitment of the challenger to select the platform with the highest poll result, provided this platform has a height of at least  $\frac{1}{2}$ .

The interest group's members voting behaviour is coordinated as follows. When the platforms of the two political parties are known the active interest groups decide which party to support and then all members of the interest group vote for that party (if a voter is a member of more than one active interest group that support different candidates, it follows the interest group giving him the highest benefit  $v_{jm}$ ). Each interest group decides which party to support as follows. If exactly one of the candidates takes the interest group's position on the relevant issue, the interest group supports that party. If one candidate is closer to the interest group's position than the other candidate, the former is supported. If both candidates have the same position as the interest group, or the distance from the interest groups positions on the relevant issue is the same, the interest group members votes according to their own utility, as given by (1). Clearly, the presence of interest groups may change the electoral landscape. As a matter of fact, interest group members voting behavior changes from being consistent with preferences induced by weighted Euclidean distance to being consistent with lexicographic preferences.

During an electoral campaign, apart from the conditioned platforms where polls are financed by interest groups, the challenger is assumed to run some polls on platforms selected randomly in the issue space. It then selects that policy position which has the best polling result. All voters organized in interest groups vote for the party

supported by the interest group, voters that belong to more than one interest group follow the interest group with the highest value of (3), all other voters vote according to the weighted euclidean distance from the different policy positions to their ideal points, as in (1). The party with the majority of votes wins the election.

### 3 Mean dynamics

The adaptive electoral competition model described in Section 2 generates a different election dynamics for each realization of ideal points and strengths and hence depends critically upon the initial configuration of the population of voters. For extensive simulations of this model and a number of extensions we refer to Sadiraj, Tuinstra and van Winden (2004). In the present paper we will focus on deriving some analytical results that may explain the outcome of these simulations. In particular, we will perform a mean-field approximation to this stochastic system, the so-called *mean dynamics*, which generates a stationary Markov process. The stationary states and the dynamics of the Markov process, which actually amounts to what happens *in expectation* in the model, can be studied and give useful information about what happens in the stochastic electoral competition models with all their individual diosyncrasies. The Markov process therefore can be viewed as the skeleton of the original process.

#### 3.1 Electoral competition as a Markov process

We have a population of  $N$  voters, with ideal positions  $x_j$  drawn from the uniform discrete distribution on  $\mathcal{X} = \{1, 2, \dots, K\} \times \{1, 2, \dots, K\}$ , where  $K$  is some odd number (which implies that the ‘center’ of the issue space  $C = (C_1, C_2) \equiv (\frac{K+1}{2}, \frac{K+1}{2})$  is, in fact, an element of  $\mathcal{X}$ ). Furthermore, voters have strengths  $s_j = (s_{j1}, s_{j2}) \in \mathcal{S} \times \mathcal{S}$ . We assume that strengths are identically and independently drawn from a discrete distribution  $p$  on  $\mathcal{S}$ , and therefore,  $p_s = \Pr(S_{j1} = s_{j1}, S_{j2} = s_{j2}) = \Pr(S_{j1} = s_{j1})\Pr(S_{j2} = s_{j2}) = p_{s_1}p_{s_2}$ . Also observe that the distribution of strengths is independent of ideal positions.

Let  $y^{t-1} \in \mathcal{X}$  be the incumbent’s platform for the election at time  $t$ , that is  $y^{t-1}$  is the winning platform of the election at time  $t - 1$ . In the following definition we define a new state space.

**Definition 1** Let  $\mathcal{R} = \{R : \exists i_1, i_2 \in \{1, \dots, K\} \text{ s.t. } R^2 = i_1^2 + i_2^2\}$ . Now define by  $\mathcal{U} = \{U_R, R \in \mathcal{R}\}$ , the family of subsets of  $\mathcal{X}$  with elements

$$U(R) = \{x \in \mathcal{X} : (x_1 - C_1)^2 + (x_2 - C_2)^2 = R^2\}.$$

The idea behind using  $\mathcal{U}$  should be clear. We are not so much interested in the actual platforms that have been selected but more in the distance of these platforms from the center  $C$  of the distribution. Moreover, due to symmetry, all platforms that are equally distant from  $C$  (i.e. that belong to the same element  $U_R$ ) can be treated similarly. Notice that the number of elements of  $\mathcal{U}$  is  $n = \sum_{k=1}^{\frac{1}{2}(K+1)} k$ , which clearly is much smaller than  $|\mathcal{X}| = K^2$ . Moreover, it is easily checked that each element of  $\mathcal{U}$

contains 1, 4 or 8 elements of  $\mathcal{X}$ . The following proposition claims that the electoral dynamics corresponds to a Markov chain on  $\mathcal{U}$ .

**Proposition 2** *The family  $\mathcal{U}$  satisfies the following properties.*

- i) It forms a partition for the space  $\mathcal{X}$ .*
- ii) For all  $R$  and  $R'$  and for all  $y^t, (y^t)' \in U_R$ ,*

$$\Pr(y^{t+1} \in U_{R'} \mid y^t) = \Pr(y^{t+1} \in U_{R'} \mid (y^t)').$$

**Proof.** Straightforward. ■

According to the second property of  $\mathcal{U}$ , the probability of moving from any platform  $z$  in  $U_R$  to platforms in  $U_{R'}$  is independent of the particular platform  $z$ .

The electoral competition now corresponds to a Markov process with stationary transition probabilities on  $\mathcal{U}$ . The next step is to derive the transition matrix for this Markov process. Now denote the  $n$  elements of  $\mathcal{U}$  as  $\{U_1, U_2, U_3, \dots, U_n\} \equiv \{U(0), U(1), U(\sqrt{2}), \dots, U(\sqrt{2}K)\}$ . We can then, for given political institutions, voter preferences and the interest group formation process, compute an  $n \times n$  transition matrix  $P_r$ , where the index  $r$  gives the number of polls. Element  $(i, j)$  of  $P_r$  gives the probability that, when the incumbent is in  $U_i$ , the election outcome will be in  $U_j$ . Notice that the randomness in the system is, given the initial position of the incumbent, the transition matrix  $P_r$  can give us a lot of information about the election dynamics, as will become clear shortly.

Now let the initial policy platform,  $y^0$ , follow some discrete distribution  $\pi_0$  on  $\mathcal{U}$  (where  $\pi_0$  might for example be generated by the uniform distribution on  $\mathcal{X}$ ). The electoral outcome evolves according to the transition probabilities given by  $P_r$  and at election  $t$  the distribution of policy platforms over the different states  $U_i$  is given by

$$\pi_t = \pi_0 (P_r)^t.$$

Apart from the limiting behavior and the transient dynamics of this Markov process, we are, for every time period  $t$ , interested in two particular variables namely the distance of the incumbent from the center of the issue space, and the probability for the challenger to win the election. Furthermore, given the distribution  $\pi_t$  of policy platforms at time  $t$ , the average distance between the policy platform  $y^t$  and the center  $C$  is given by

$$E(\|y^t - C\|) = \sum_{R \in \mathcal{R}} R \pi_t.$$

The probability that the challenger wins at an election  $t$  is

$$\Pr(\text{the challenger wins at time } t) = \pi_t w,$$

where  $w = (w_R)_{R \in \mathcal{R}}$  with

$$w_R = \Pr(\text{challenger wins} \mid \text{incumbent's position } y \text{ is an element of } U_R).$$

An algorithm outlining how to compute the transition matrix  $P_r$  and the vector  $w$ , for the different models can be found in the Appendix.

In the next two subsection we will investigate the mean dynamics for the benchmark model and the interest group model for one typical specification. We take  $K = 5$ , and  $\mathcal{S} = \{0, \frac{1}{2}, 1\}$ . Strengths are drawn from  $\mathcal{S}$  according to the following distribution  $\Pr(s_{ji} = 0) = \Pr(s_{ji} = 1) = \frac{1}{4}$  and  $\Pr(s_{ji} = \frac{1}{2}) = \frac{1}{2}$ , for  $j \in \{1, 2, \dots, N\}$ . Let the initial platform  $y^0$  be drawn from the uniform distribution on the issue space  $\mathcal{X} = \{1, \dots, 5\} \times \{1, \dots, 5\}$ . The statepace for the Markov process in this case becomes

$$\mathcal{U} = \left\{ U_R \mid R \in \left\{ 0, 1, \sqrt{2}, 2, \sqrt{5}, 2\sqrt{2} \right\} \right\}.$$

and, under the assumption that  $y^0$  is drawn from a uniform distribution on  $\mathcal{X}$  we have  $\pi_0 = \left[ \frac{1}{25}, \frac{4}{25}, \frac{4}{25}, \frac{4}{25}, \frac{8}{25}, \frac{4}{25} \right]$ .

### 3.2 Dynamics for the benchmark model

Using the algorithm given in the appendix, we have computed the transition matrices  $P_r$  and the vector  $w_{(r)}$  of probabilities with which the challenger is expected to win, for two different values of the number of random polls,  $r$ :

- $r = 2$ ,

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.080 & 0.920 & 0 & 0 & 0 & 0 \\ 0.080 & 0.287 & 0.633 & 0 & 0 & 0 \\ 0.077 & 0.270 & 0.253 & 0.400 & 0 & 0 \\ 0.073 & 0.273 & 0.248 & 0.175 & 0.230 & 0 \\ 0.07 & 0.253 & 0.237 & 0.147 & 0.273 & 0.020 \end{pmatrix}, w_{(2)} = \begin{pmatrix} 0.50000 \\ 0.53999 \\ 0.68333 \\ 0.80000 \\ 0.90833 \\ 0.99000 \end{pmatrix}$$

and

- $r = 10$ ,

$$P_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.400 & 0.600 & 0 & 0 & 0 & 0 \\ 0.400 & 0.543 & 0.057 & 0 & 0 & 0 \\ 0.250 & 0.495 & 0.253 & 0.002 & 0 & 0 \\ 0.152 & 0.533 & 0.308 & 0.006 & 0.001 & 0 \\ 0.090 & 0.407 & 0.422 & 0.001 & 0.080 & 0 \end{pmatrix}, w_{(10)} = \begin{pmatrix} 0.50000 \\ 0.70000 \\ 0.97174 \\ 0.99878 \\ 1.0000^* \\ 1.00000 \end{pmatrix}$$

where 1.0000\* refers to 0.999998 which has been set to 1 in rounding. Let  $P_r(i, j)$  be the element in the  $i$ -th row and  $j$ -th column of  $P_r$ . Then  $P_r^n(i, i) = [P_r(i, i)]^n$ , since  $P_r$  is a lower diagonal matrix. Hence, for all  $i = 2, \dots, 6$ ,  $\sum_n P_r^n(i, i) < \infty$  as a geometric series with term  $|P_r(i, i)| < 1$ . Thus all states  $U_R, R > 0$  are transient since from the theory of Markov Chains, transience (persistence) of a state  $j$  is equivalent to  $\sum_n P^n(j, j) < \infty$  ( $= \infty$ ) (see Billingsley (1986, p.114)). Furthermore,  $P_{11} = 1$  implies

that  $\{U_0\}$  is a closed<sup>2</sup> set and  $U_0$  a persistent state. Thus, we have that the stationary distribution is  $\pi^* = [1, 0, 0, 0, 0, 0]$ , and in the long run: (i) the policy platform will end up in the center position  $C$  and stay there forever; and (ii) the probability the challenger wins at an election converges to 0.5 ( $= \lim_{t \rightarrow \infty} \pi_t w = \pi^* w = w_0$ ).<sup>3</sup>

### 3.3 Dynamics for the model with interest groups

Interest groups influence the elections process in three ways: (i) they coordinate the voting behavior of their members; (ii) they provide information about the electoral landscape to the political parties; and (iii) they try to influence the policy outcome via conditions on polling. In order to be able to disentangle the impact of the latter from the first two we present the dynamic analysis of the model with interest groups for two different polling procedures: ‘conditional’ polling and ‘unconditional’ polling.

#### 3.3.1 Unconditional polling

We start with recalling how the voting behaviour of an interest group’s members is coordinated. When the platforms of the two political parties are known the active interest groups decide which party to support and then all members of the interest group vote for that party (if a voter is a member of two active interest groups that support different candidates, then that voter is assumed to follow the interest group with the highest benefit  $v_{jm}$ , as in the previous section). Each interest group decides which party to support as follows. If exactly one of the candidates takes the interest group’s position on the relevant issue, the interest group supports that party. If one candidate is closer to the interest group’s position than the other candidate, the former is supported. If both candidates have the same position as the interest group, or the distance from the interest group positions on the relevant issue is the same, the interest group members votes according to their own utility, as given by (1). Our first research question is to investigate the effects of the new (if any) properties of the electoral landscape in the dynamics of the electoral outcomes. For this we assume that the challenger runs  $r$  random polls. It should be clear by now that this case is exactly the same as the basic one, corrected for the strength profiles of interest group members changing from  $s$  to  $(1, 0)$  or  $(0, 1)$ . The transition matrix,  $P_{10}^I$  and the vector,  $w_{10^u}^I$  of winning probabilities for the model with interest groups, turns out to be

$$P_{10^u}^I = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 \\ 0.152 & 0.848 & 0 & 0 & 0 & 0 \\ 0.400 & 0.425 & 0.176 & 0 & 0 & 0 \\ 0.007 & 0.443 & 0.407 & 0.142 & 0 & 0 \\ 0.152 & 0.444 & 0.307 & 0.007 & 0.090 & 0 \\ 0.028 & 0.407 & 0.542 & 0^* & 0.023 & 0 \end{pmatrix}, w_{(10^u)}^I = \begin{pmatrix} 0.50000 \\ 0.99878 \\ 0.99985 \\ 1.00000 \\ 1.00000 \\ 1.00000 \end{pmatrix}$$

<sup>2</sup>A set  $B$  in  $S$  is *closed* if  $\sum_{j \in B} P(i, j) = 1$  for  $i \in B$ : once the system enters  $B$  it cannot leave (see Billingsley (1986, p.140)).

<sup>3</sup>Recall that we have assumed that, if the challenger does not find a platform with  $h(z | y) > 0.5$ , it chooses the incumbent’s platform  $y$ , and wins with probability 0.5.

where  $0^*$  refers to 0.00002, which has been set to 0 in rounding. As for the basic model, we find that there is one and only one closed set, the elements of which are all persistent states, which is  $\{U_0\}$ . All states  $U \in \mathcal{U} \setminus U_0$  are transient. However, there is a difference in the speed with which the system convergence to the center as the following shows. Figures 1 and 2 give, for the 3 different cases, diagrams with  $E(\|y^t - C\|)$  and  $\Pr$  (the challenger wins at time  $t$ ), respectively. First consider Figure 1. From the highest to the lowest curve we have: benchmark model with 2 random polls, interest group model with 10 random polls, benchmark model with 10 random polls. We can draw the following conclusions from this figure. Firstly, an increase in the number of (unconditional) polls decreases the expected separation between the winning platform and the center of the distribution. Secondly, for the interest group model expected separation is larger than for the basic model with the same number of polls. For Figure 2 the highest to the lowest curve (as measured at election 6) are respectively: the interest group model with 10 random polls, the basic model with 10 polls and the basic model with 2 polls. From this it follows that the presence of interest groups increases the probability of winning an election. One of the findings in Sadiraj, Tuinstra and van Winden (2004) was that the presence of interest groups appears to increase the winning set. That result is confirmed here as well. Given the state of the incumbent, we find that the size of the winning set equals: (a)  $(0 \ 1 \ 5 \ 9 \ 14 \ 21)'$  for the basic model, and (b)  $(0 \ 9 \ 11 \ 17 \ 19 \ 22)'$  for the model with interest groups (recall that  $|\mathcal{X}| = 25$ ). Note that since  $\{U_0\}$  is a closed set, if the incumbent is in  $U_0 = \{C\}$  then the winning set is empty 0. These figures show that except for the case in which the incumbent's platform is in  $U_0$ , the size of the winning set increases in the presence of interest groups.<sup>4</sup>

Time series of the expected probabilities with which the challenger defeats the incumbent.

### 3.3.2 Conditional polling

As mentioned above, the interest groups influence the election process by providing information about the electoral landscape to the political parties. Let us recall what information they provide. Each interest group possesses certain funds raised by the contributions  $c$  of its members. These funds are offered to the challenger conditional on: *i*) running a number of polls<sup>5</sup> in policy positions coinciding with the interest group's position on the relevant issue; *ii*) commitment of the challenger to select the platform with the highest poll result, if this platform has a height of at least  $\frac{1}{2}$ . Furthermore, it is assumed that each interest group knows the median of the distribution of its group's members on the other issue and finances a poll there. Let  $r_1$  be the number of random polls and  $r_2$  the number of conditioned polls. Let the challenger first run  $r_2$  conditioned polls and then  $r_1$  random polls. Removing from the policy space the positions where the conditioned polls are run, and using formulas (5),

<sup>4</sup>The result is robust to changes in all parameter settings we have investigated. We have derived similar results for different distributions  $p$  on  $\mathcal{S}$ , and different number of positions per issue ( $K \in \{3, \dots, 11\}$ ).

<sup>5</sup>Remember that the number of polls that an interest group can finance is determined by the cost of running a poll and the size of the fund that the group possesses.

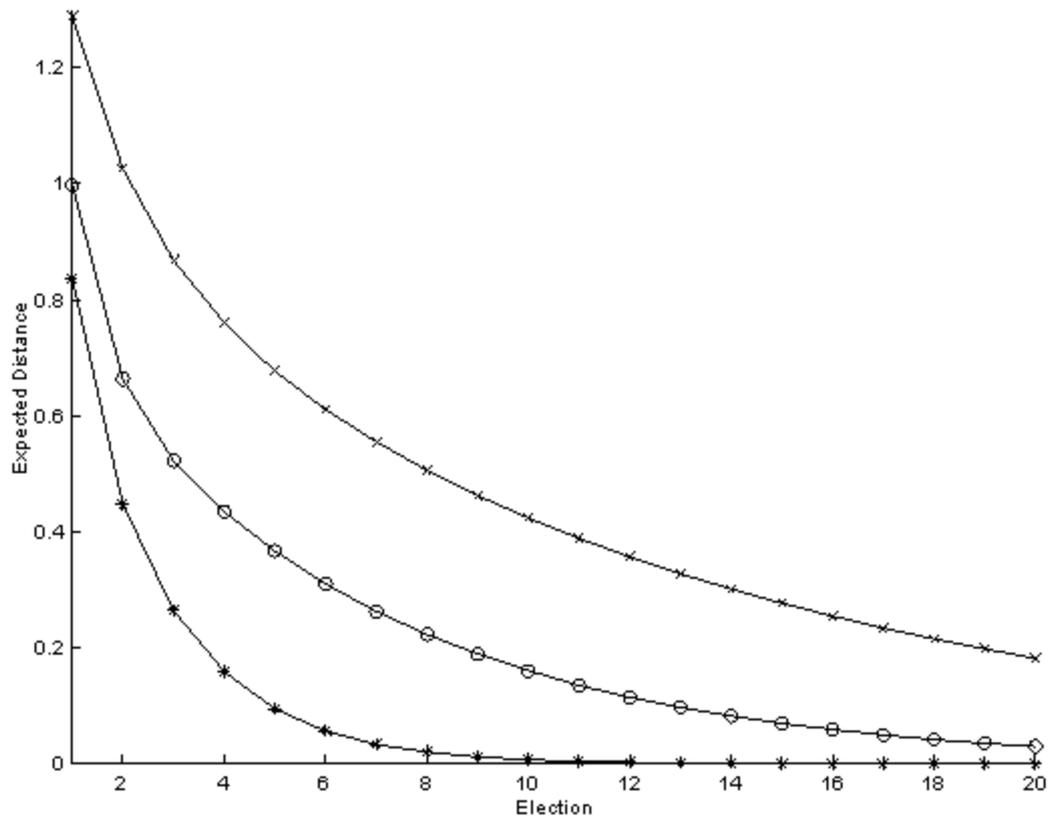


Figure 1: Time series of the expected distance between the incumbent and the center of the space.

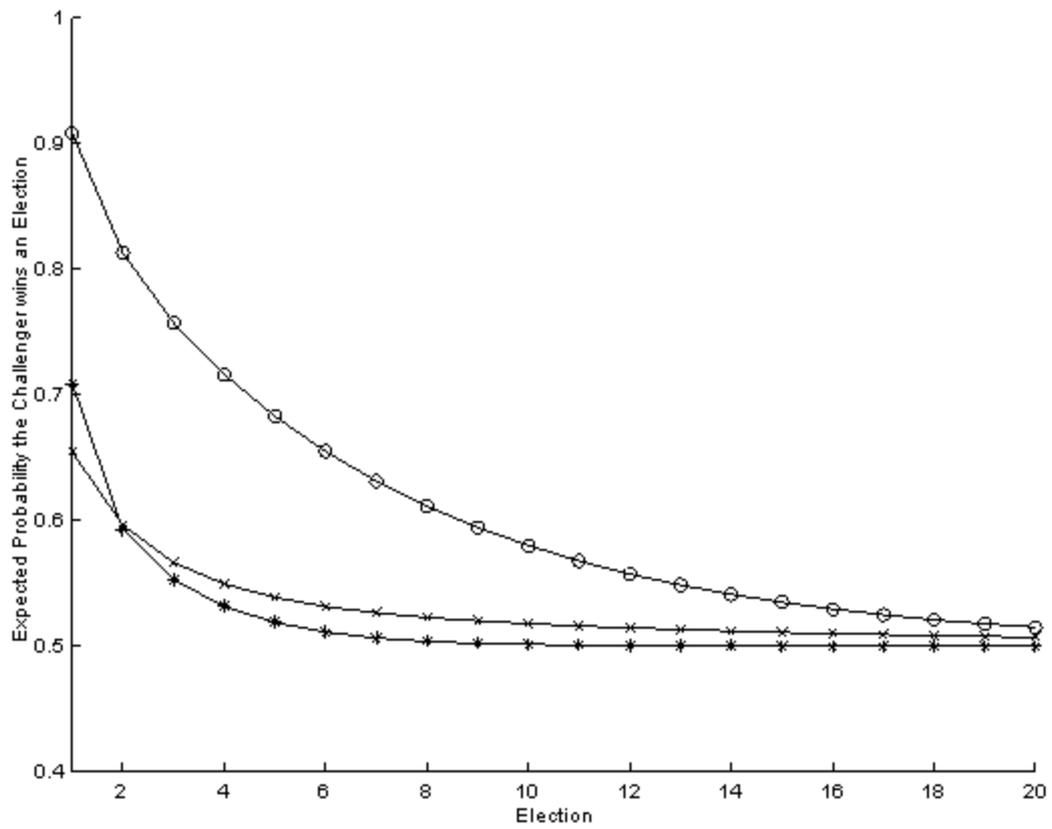


Figure 2: Time series of the expected probabilities with which the challenger defeats the incumbent.

|                |   |                |      |      |      |      |
|----------------|---|----------------|------|------|------|------|
|                | 5 | ×              | ×    | .538 | ×    | ×    |
| <i>Issue 2</i> | 4 | ×              | ×    | .590 | .538 | .508 |
|                | 3 | ×              | .500 | .575 | .500 | ×    |
|                | 2 | ×              | ×    | .590 | .538 | .508 |
|                | 1 | ×              | ×    | .538 | ×    | ×    |
|                |   | 1              | 2    | 3    | 4    | 5    |
|                |   | <i>Issue 1</i> |      |      |      |      |

Table 1: Fractions of voters who prefer a position  $z = (i, j)$  to  $(2, 3)$  ( $\times$  refers to fractions less than 0.5).

(7) and (8), one can compute the transition probabilities for the conditional polling procedure.

For the specified model and  $r_2 = 8, r_1 = 2$ , we find

$$P_{10^c}^I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.882 & 0.118 & 0 & 0 & 0 \end{pmatrix}, w_{10^c}^I = \begin{pmatrix} 0.5 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

A new persistent state appears. In addition to state  $U_0$  which remains a persistent state with the property ‘ $\{U_0\}$  is a closed set’, state  $U_1$  becomes a persistent state as well with the property ‘ $\{U_1\}$  is a closed set’. This can be derived as follows. The transition matrix shows that if the system at election  $t$  is in one of the states  $U_R, R \in \{1, 2, \sqrt{5}\}$ , then at election  $t + 1$  it will be in  $U_1$  and stay there forever. If the system starts at  $U_{2\sqrt{2}}$  then, with probability 0.882, in the coming election it will end up in  $U_1$  and never leave that state. The probability that the system will settle in  $U_1$  is given by the first coordinate of  $\pi_0 P_{10^c}^I$  and equals 0.781. In the same way one can derive that the system will settle in  $U_0$  with probability 0.219. Furthermore, let the incumbent platform be  $y = (2, 3) \in U_1$ .<sup>6</sup> Table (1) shows the fraction of votes that the challenger gets if he selects a position  $z = (i, j), i, j = 1, \dots, 5$ , ( $\times$  refers to fractions of votes smaller than 0.5). Thus, the winning set that corresponds to a position  $y \in U_1$  has always at least two elements from  $U_1$  with the highest fraction of votes. Let us now consider the interest group located at position 2 on the second issue. From the uniformity of the distribution of voters in the space and the homogeneity<sup>7</sup> of voters within types, it follows that the median of the members of this interest group related to the first issue is located at 3. Hence, that interest group will finance a poll at position  $(3, 2)$ . Note that the altitude at  $(3, 2)$  is .59, which is the highest value in Table (1). Thus, the incumbent platform in the coming election will be either  $(3, 2)$  or  $(3, 4)$ . This means that although the incumbent does not leave the  $U_1$  set,

<sup>6</sup>It should be clear (for reasons of symmetry) that Table (1) for a  $y \in U_1$  is the same as the one derived by rotating Table (1) around the center  $(3, 3)$  until  $(2, 3)$  reaches  $y$ .

<sup>7</sup>Voters of some type  $s$  and with the same ideal positions on some issue  $i$ , make the same decisions to join the relevant interest group.

a voting cycle appears. Therefore we may conclude that, with probability .781, (i) a cycle emerges and (ii) the challenger wins with probability 1.

## 4 Voting cycles driven by interest groups

The ‘mean dynamic’ analysis from the previous section shows that for the specified parameters of the models, there is only one closed set,  $\{U_0\}$  in the basic case. However, under conditional polling, there are two closed sets,  $\{U_0\}$  and  $\{U_1\}$ , for the model with interest groups. This raises the question of the dependence of this result on the parameter specification, like the size of the space, the set of strengths, the probability distribution of strengths on that set and so on. The following analysis provides an answer to that question.

**Proposition 3** *Assume voters’ ideal positions are independently (across issues and across voters) drawn from the uniform distribution on  $\mathcal{X}$  and voters’ strengths are independently drawn from a discrete distribution on  $\mathcal{S}$  and are uncorrelated with the ideal positions. Denote by  $y = (y_1, y_2)$  the platform of the incumbent. We then have:*

1. (a) For both models (with and without interest groups),  $Z_{0.5}^+(C) = \emptyset$  and  $Z_{0.5}^0(C) = U_0$ .  
 (b) For the basic model,  $\forall y \in \mathcal{X} \setminus \{C\}$ ,  $C \in Z_{0.5}^+(y)$ ;
2. If the distribution of strengths satisfies

$$\frac{(1 - \frac{1}{K}) \sum_{s \in \mathcal{S} \setminus \{0\}} p_s^2 + (\frac{3}{2} - \frac{1}{K}) \sum_{(s_1/s_2)=2} p_{s_1} p_{s_2}}{\sum_{s \in \mathcal{S} \setminus \{0\}} p_s} > \frac{1}{2} \quad (4)$$

then in the presence of interest groups,  $\forall y \in U_1, \exists y' \in U_1 \setminus \{y\}$ , such that  $h(y' | y) > h(z | y)$ , for all  $z \in \cup_{R \in \mathcal{R} \setminus \{1\}} U_R$ .

**Proof.** see the Appendix ■

**Corollary 4** *For the specified models, Proposition 3 applies.*

**Proof.** Indeed, substituting  $K = 5$ ,  $\sum_{s \in \mathcal{S} \setminus \{0\}} p_s^2 = (1/2^2 + 1/4^2)$ ,  $\sum_{(s_1/s_2)=2} p_{s_1} p_{s_2} = 1/8$ , and  $\sum_{s \in \mathcal{S} \setminus \{0\}} p_s = 3/4$  at the lhs of the inequality (4), we find a value of 0.55 which is bigger than 0.5 which is the value of the rhs of (4). ■

In words, Corollary 4 shows that for the specified models, analyzed in Section 3, the following properties hold: (i) 3.1(b) implies that for the basic model, for any incumbent position  $y$ , different from  $C$ , position  $C$  has an altitude larger than 0.5, and hence belongs to the winning set of  $y$ ; (ii) 3.1(a) implies that for both models, if the incumbent is at  $C$ , then any position that is not in  $U_0$  has an altitude strictly less than 0.5, and hence it does not belong to the winning set of  $C$ ; and (iii) 3.2 implies that in the presence of interest groups, there is at least one position  $y'$  in  $U_1$  which is different from  $y$  and has altitude larger than any other position that does not belong to  $U_1$  provided that property (4) is satisfied.

**Proposition 5** 1. For both models, with and without groups,  $\{U_0\}$  is a closed set and  $U_0$  is a persistent state,

2. For the model without interest groups, all other states,  $U \in \mathcal{U} \setminus U_0$  are transient,

3. Assume that the distribution of strengths has property (4). In the presence of interest groups and given conditional polling,

(a)  $\{U_1\}$  is a closed set and  $U_1$  is a persistent state, and

(b) voting cycles emerge once the incumbent visits  $U_1$ .

Corollary 4 implies that the Proposition 5 is relevant for the specified models. Hence, the results shown in Section 3 can be easily derived by applying Proposition 5.

We conclude that our models have all the properties presented in Proposition 5 for all specifications of parameters that satisfy condition (4).

## 5 Simulations and empirical illustration

The law of large numbers ensures us that the mean-analysis is relevant for populations that are large enough to correct for random deviations. However, the population of voters may not be large enough to cancel out random fluctuations, and therefore, the law of large numbers may not apply. This may have consequences at the macrolevel. That is why in this section we will consider some simulations for different realizations of voter preferences and investigate whether the predictions of Proposition 5 are valid. Furthermore, we will compare these simulation results to some empirically observed policy outcomes.

Each trial starts with drawing a population of 1000 voters from the uniform distribution on  $\mathcal{X}$ , where we again assume  $K = 5$ . The initial position of the incumbent is chosen to be the center, in order to be able to investigate the closeness property (see footnote 3) of this center for the different models. Each trial was run for 20 elections and we have done 20 different trials. Typical results are represented in panels (a)-(d) of Figure 3. Panels (a) and (b) show that in the basic model the incumbent remains at the center for all elections. This is a robust feature of all trials with the basic model. Panels (c) and (d) of Figure 3 show that in the interest group model something different occurs: counter to the first statement in Proposition 5, the incumbent leaves the center and positions itself at some other position. This happens in more than half of all trials.

From these figures it is apparent that for the basic model, the set that contains the center of the issue space,  $\{C\}$ , is a closed set even for the stochastic model. However, for the interest group model, the center loses that property for certain realizations of the distribution of voter preferences. For our issue space of 25 positions, simulations show that: for the basic model the property that  $\{U_0\}$  is a closed set is maintained if the size of the population is larger than 300; for the model with interest groups,  $\{U_0\}$  and  $\{U_1\}$  are closed sets if the size of the population is larger than 10000.

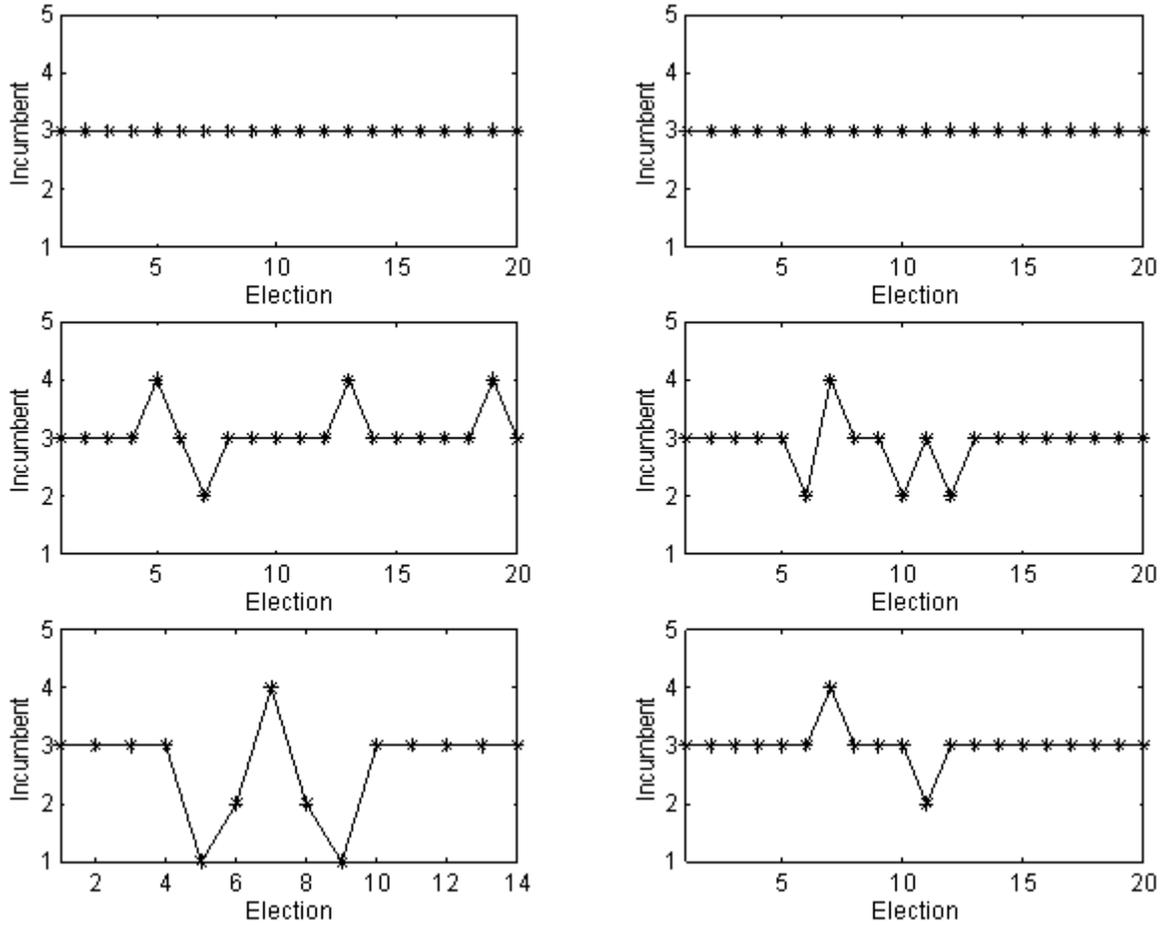


Figure 3: The stability of the center:incumbent and empirical data. Panels (a) and (b) show data generated from the benchmark model in simulations 13 and 14, respectively. Panels (c) and (d) show data generated from the model with interest groups in simulations 13 and 14, respectively. Panels (e) and (f) show data generated from the composition of the governments in Finland and Iceland, respectively.

For populations with size smaller than 1000, neither  $\{U_0\}$  nor  $\{U_1\}$  are closed sets. Our next step is to relate these simulation results to some empirical data on policy outcomes. An analysis of the policy outcomes for 20 European countries was done in Woldendorp, Keman and Budge (1998). They classified the composition of the government as falling into one of 5 categories, ranging from extreme left (category 1) to extreme right (category 5). The graphs represented in panels (e) and (f) of Figure 3, respectively, correspond to Iceland data and Finland data, starting with the first time the composition of the government is in the center (position 3) after 1960. We draw attention to two features present in the data from both countries: (i) the government composition stays longer at position 3 than at the other positions, that is, the center presents a position which is hard to be defeated from other positions; (ii) although the government composition locates at 3 it does not stay there forever, that is, the center can be defeated. Comparing these graphs to the graphs generated by the simulations it is clear that the data generated by the interest group model represents the empirical data best. In our view, this may provide some support for the model with interest groups presented in Section 2.

## 6 Concluding remarks

Although simulations provide a valuable aid in characterizing the system's behavior, their power is limited to the domain of the selected parameters. An understanding of the more generic properties of individual-based models requires the use of deterministic approximation models. In this paper we have applied a mean-field approximation to the stochastic models presented in Section 2, by replacing the values of the random variables by their expected values. This leads to deterministic dynamic models of the "Markov" type. The main results obtained from the analysis of the deterministic models are as follows. The dynamics of the distance between the policy outcome and the center of the space, and of the probability that the challenger wins an election, replicate qualitatively the respective dynamics generated by the individual-based models. For both models, with and without interest groups, the set consisting of the center of the space presents a closed set. For a certain class of probability distributions on a set of strengths  $\mathcal{S}$  and under conditional polling, it is shown that (i) the set of positions at distance 1 from the center is a closed set for the model with interest groups, and (ii) a voting cycle emerges. For the specified model the voting cycle appears with probability .781. Simulations show that for populations of size smaller than 1000 neither the center nor the 4 positions (at distance 1) closest to the center present closed sets for the model with interest groups. Voting cycles become frequent phenomena and expand all over the issue space,<sup>8</sup> due to the deviations of the realizations of the random variables (the distributions of voters preferences) from the mean values. To our knowledge, this is the first study pointing at, and providing a micro-foundation for, the possibility of a voting cycle in the presence of a dominant point. Although for different reasons, our work contributes to research in the spatial theory of elections that advocates that "...democratic voting is characterized by

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<sup>8</sup>Interested readers can contact the author for animations that illustrate this behaviour.

forces that keep outcomes reasonably close to the center of voter opinion... ” (Enelow and Hinich (1984, p. 223)). A further investigation shows however, that if the size of the population is lower than some threshold (1000 for our specified model) voting cycles become frequent phenomena and expand all over the issue space. Thus for small populations, our model positions itself in the series of models that point at the electoral instability of voting outcomes (McKelvey (1976, 1979), Schofield (1978)).

The inherent property driving our results is that the winning set (i.e., the set of policy platforms that will defeat the current incumbent) increases in the presence of interest groups. This happens in all the stochastic and numerical simulations. Moreover, in Sadiraj, Tuinstra and van Winden (2005b) it is rigorously shown that, in a slightly different spatial competition framework and under certain mild conditions on the incumbent’s position, the winning set for the challenger indeed increases when interest groups are present to coordinate voting behavior.

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## Appendix

### An algorithm for determining the transition matrix

Let  $y^t$  be the platform of the incumbent after election  $t$ . In order to compute the transition probabilities, first we need to compute the height of the electoral landscape for any position  $z$  in the issue space, given the incumbent's platform  $y^t$ , i.e. we need  $h(z | y^t)$  for all  $z$ .

1. Denote by  $u(z | x, s)$  ( $= -\sum_{i=1}^2 s_i(x_i - z_i)^2$ ) the utility that a voter with ideal position  $x$  and vector of strengths  $s$  derives from the policy  $z$  when implemented. For each position  $z \in \mathcal{X}$ , the height of the electoral landscape  $h(z | y^t)$  is given by the following formula

$$h(z | y^t) = \sum_{x \in \mathcal{X}, s \in \mathcal{S}} [\Pr(X = x, S = s) G(u(z | x, s) - u(y^t | x, s))], \quad (5)$$

where

$$G(v) = \begin{cases} 1 & v > 0 \\ \frac{1}{2} & v = 0 \\ 0 & v < 0 \end{cases}. \quad (6)$$

In words, the above formula can be described as: first, take a position  $x$  and count with weight  $1(1/2)$  voters with ideal position  $x$  that strictly prefer (are indifferent between) policy  $z$  to  $y^t$ ; next, take the weighted sum of the above figures for all  $x \in \mathcal{X}$ . The outcome gives the fraction of votes that the challenger is expected to get if he selects  $z$  given that the incumbent platform is  $y^t$ .

2. Define

$$\begin{aligned} Z_l^0(y) &= \{z \in \mathcal{X} \mid h(z \mid y) = l\}, \\ Z_l^+(y) &= \{z \in \mathcal{X} \mid h(z \mid y) > l\}, \\ Z_l^-(y) &= \{z \in \mathcal{X} \mid h(z \mid y) < l\}. \end{aligned}$$

That is, the set  $Z_l^0(y)$  ( $Z_l^+(y)$ ,  $Z_l^-(y)$ ) contains all positions for which, given the platform of the incumbent, the height of the electoral landscape is equal to (larger than, smaller than)  $l$ . Notice that these sets are disjoint and  $Z_l^0(y) \cup Z_l^+(y) \cup Z_l^-(y) = \mathcal{X}$  for all  $l$  and  $y$ . Obviously,  $y^{t+1} \in Z_{0.5}^0(y^t) \cup Z_{0.5}^+(y^t)$  and the set  $W(y^t)$  of positions that defeat the incumbent at  $y^t$  is given by  $W(y^t) = Z_{0.5}^+(y^t)$ .

Let the challenger run  $r$  polls. Denote by  $T \subset \mathcal{X}$  the set of positions that is polled by the challenger. The transition probabilities can now be computed as follows. First, we compute the probability  $p_l$  that the challenger selects a position from the set  $Z_l^0(y)$ . That event happens if the challenger runs: (a) at least one poll at a position  $z$  from  $Z_l^0(y)$  and (b) no polls at positions from  $Z_l^+(y)$ . That is

$$p_l = \Pr(T \cap Z_l^0(y) \neq \emptyset \text{ and } T \cap Z_l^+(y) = \emptyset) \quad (7)$$

which, in case of random polling becomes

$$\begin{aligned} p_l &= \sum_{k=1}^r \binom{r}{k} \Pr(|T \cap Z_l^0(y)| = k) \Pr(|T \cap Z_l^-(y)| = r - k) \\ &= \sum_{k=1}^r \binom{r}{k} \frac{\prod_{i=0}^{k-1} \max\{|Z_l^0(y)| - i, 0\} \prod_{j=0}^{r-k-1} \max\{|Z_l^-(y)| - j, 0\}}{\prod_{m=0}^{r-1} (K^2 - m)}. \end{aligned}$$

Thus, in other words we have derived the probability that the highest altitude in the polling process equals  $l$ .

Next, define  $L(y, R') \equiv \{l \geq \frac{1}{2} \mid U_{R'} \cap Z_l^0(y) \neq \emptyset\}$ . The transition probabilities then can be computed as

$$\Pr(U_{R'}, U_R) = \Pr(y^{t+1} \in U(R') \mid y^t \in U(R)) = \sum_{l \in L(y^t, R')} \frac{p_l}{|Z_l^0(y^t)|} G\left(l - \frac{1}{2}\right), \quad (8)$$

with  $G$  as defined in (6).

The probabilities given by (8) define a (stationary)  $|\mathcal{U}| \times |\mathcal{U}|$  transition matrix  $P_r$ , where  $|\mathcal{U}|$  denotes the cardinality of the set  $\mathcal{U}$

For example, for the benchmark model we have  $\Pr(X = x) = \frac{1}{K^2}$  and  $\Pr(S = s) = p_{s_1} p_{s_2}$  as given before. For these specifications (5) becomes

$$h(z \mid y^t) = \sum_{s \in \mathcal{S}, x \in \mathcal{X}} \frac{p_s}{K^2} G(u(z \mid x, s) - u(y^t \mid x, s)).$$

Election dynamics of the Markov chain

Let the initial platform,  $y^0$  be drawn from a discrete distribution  $\pi_0$  on  $\mathcal{U}$ . The electoral outcome evolves according to the above transition probabilities and at election  $t$  we have that the incumbent is expected to be in one of the states  $U_R$  according to the following distribution

$$\pi_t = \pi_0 (P_r)^t.$$

Furthermore, at each election  $t > 0$ , the incumbent is expected to be at distance  $d(y^t, C)$  from the center of the distribution of voter preferences given by

$$E(\|y^t - C\|) = \sum_{R \in \mathcal{R}} R \pi_t.$$

The probability that the challenger wins at an election  $t$  is

$$\Pr(\text{the challenger wins at time } t) = \pi_t w,$$

where  $w = (w_R)_{R \in \mathcal{R}}$  with

$$w_R = \frac{1}{2} \Pr(y^{t+1} \in Z_{0.5}^0(y^t) \mid y^t \in U_R) + \Pr(y^{t+1} \in Z_{0.5}^+(y^t) \mid y^t \in U_R).$$

### Proof of Proposition 3

1. Let a position  $z \neq C$  be given. Denote by voters of type  $s$  all voters with a certain strength vector  $s \in \mathcal{S} \times \mathcal{S}$ . Consider first the basic model. For each strength profile  $s \in \mathcal{S} \times \mathcal{S}$ , we can identify the line  $l_s$  of ideal points, going through  $(z + C)/2$ , for which voters of type  $s$  will be indifferent between  $z$  and  $C$ . Each of those lines separates the space into two disjoint subspaces, one of which contains  $z$  and the other contains  $C$ , unless both  $C$  and  $z$  belong to  $l_s$ . The uniform distribution of ideal points implies that for all  $s$ , if  $C \notin l_s$  ( $C \in l_s$ ) then the subspace where  $C$  lies contains more than (at least) half of the voters of type  $s$ , and all of them by construction prefer (are indifferent between)  $C$  to  $z$ . Since the set of strengths has at least two elements and  $z \neq C$  there is at least one  $l_s$  such that  $C \notin l_s$  and thus in expectations, while competing with  $C$ ,  $z$  gets less than half of the votes, i.e.  $z \notin Z_{0.5}^+(C) \cup Z_{0.5}^0(C)$ . By assumption the challenger and the incumbent share votes if they adopt the same position and hence,  $Z_{0.5}^+(C) = \phi$ , and  $Z_{0.5}^0(C) = \{C\}$ . Thus, 3.1(a) for the basic model is shown. Furthermore, since more than half of the voters prefer  $C$  to  $z$ ,  $C \in Z_{0.5}^+(z)$ , 3.1(b) is proven.

Consider now the model with interest groups. To show that for all positions  $z$  such that  $z \neq C$ , the property  $z \notin Z_{0.5}^+(C)$  holds we do the following. Without loss of generality (due to the symmetry in the model) let  $z_i \leq C_i$ ,  $i = 1, 2$ . First, for all  $s \in \mathcal{S} \times \mathcal{S}$ , draw lines  $l_-$  and  $l_+$  through  $C$  as defined in Definition 6 from Sadiraj, Tuinstra and van Winden (2005b). Next, using Lemma 3 from Sadiraj, Tuinstra and van Winden (2005b) note that for the given  $z$ : (i) there

can be no voters of type  $s$  with ideal positions either to the right of  $l_-$  or at  $C$  that would prefer  $z$  to  $C$ ; and that (ii) voters of type  $s$  with ideal positions on  $l_- \setminus \{C\}$  would be indifferent between joining either group and hence  $C$  and  $z$  would get each half of those voters. Finally, consider all types of voters  $s \in S$ , and properties (i) and (ii) for each of  $s \in S$ , and derive that less than half of the voters of type  $s$  would vote for  $z$  versus  $C$ , i.e.  $z \notin Z_{0.5}^+(C) \cup Z_{0.5}^0(C)$ . That  $C \in Z_{0.5}^0(C)$  is clear.

2. Let  $y \in U_1$  and a position  $z \in \cup_{R \neq 1} U_R$  be given. To fix the idea, let  $y = (C_1, C_2 + 1)$  and  $z_1 \leq y_1$ . First we calculate  $h(C | y)$ . For this, note that the center  $C$  would get votes from voters of type  $s \in \bar{S} = \{s \in \mathcal{S} \times \mathcal{S}, s_2 \neq 0\}$ , with ideal positions below the horizontal line going through the point  $m_y C = (C_1, \frac{y_2 + C_2}{2})$ . Since there are  $K(K+1)/2$  such positions,<sup>9</sup> and since the probability that a voter has a vector of strengths  $s \in \bar{S}$  is  $\sum_{s_2 \in \mathcal{S} \setminus \{0\}} p_{s_2} = 1 - p_0$ , we derive that  $\frac{K(K+1)}{2K^2}(1 - p_0)$  is the fraction of voters who prefer  $C$  to  $y$ . Furthermore, the rest of voters would have: (a) either both strengths equal to 0, and therefore would vote randomly, or (b) the strength in the first issue  $s_1 \neq 0$  and in the second one,  $s_2 = 0$ , and since  $y_1 = C_1$  would again be indifferent between positions  $C$  and  $y$ . Thus  $\frac{p_0}{2}$  is the fraction of voters with  $s_2 = 0$  that  $C$  is expected to get. We can therefore conclude that the fraction of the votes the challenger gets if the incumbent is at platform  $y$  and if the challenger selects position  $z$  is given by

$$h(C | y) = \frac{1}{2} + \frac{1 - p_0}{2K} (> \frac{1}{2}). \quad (9)$$

Furthermore, a similar argument shows that if  $z_1 = y_1$ , then  $h(z | y) < h(C | y)$ .

Second, we show that there is a  $y' \in U_1$  such that the challenger gets a higher fraction of the voters by selecting  $y'$  rather than  $C$ . Take  $y' = (C_1 - 1, C_2)$ . Let a vector of strengths,  $s \in \mathcal{S} \times \mathcal{S}$  be given. Draw lines  $l_-, l_+$  through  $y$  as defined in Definition 6 of Sadiraj, Tuinstra and van Winden (2005b). The same reasoning as the one used in the proof of Proposition 4 from Sadiraj, Tuinstra and van Winden (2005b). However, in this case the triangles defined there correspond to empty sets and all positions to the left of  $l_-$  will vote for  $y'$ . Furthermore, all voters on positions on  $l_-$ , but different from  $y$  vote for the challenger at  $y'$  with probability  $\frac{1}{2}$ . This implies that all voters with strength vector  $s$  and ideal positions in the subspace  $V(y', y; s) = \left[1, \frac{y'_1 + y_1}{2}\right] \times [1, \min\{K, l_-\}] \cup \left[\frac{y'_1 + y_1}{2}, K\right] \times \left[1, \min\left\{\frac{y'_2 + y_2}{2}, l_+, l_-\right\}\right]$ , would vote for  $y'$  rather than  $y$ , except for those on the borders  $l_-, \frac{y'_1 + C_1}{2}$  or  $\frac{y'_2 + y_2}{2}$  who vote with probability  $\frac{1}{2}$  for the challenger. We classify  $s$  in the following disjoint sets.

- (a)  $S^a = \{s \in \mathcal{S} \times \mathcal{S}, s_1 = s_2 \neq 0\}$ . First note that for all  $s \in S^a$ ,  $l_-$  is the same. Then, using the principle of mathematical induction, we derive that

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<sup>9</sup>This can be easily shown by using the principle of mathematical induction. That is: first, show that the property holds for  $K = 3$ ; next, show that if the property holds for a  $K$ , then it does for  $K + 2$  as well.

the number of positions in  $V(y; s)$  that do not belong to  $l_-$  equals  $\frac{K(K+1)}{2}$  and the number of ones on line  $l_-$  is given by  $K - 2$ . Thus, the fraction of votes that  $y'$  is expected to get from voters of type  $s \in S^a$  is given by

$$h(y' | y, S^a) = \left( \frac{K(K+1)}{2} + \frac{K-2}{2} \right) \sum_{s \in S^a} \frac{p_s}{K^2}.$$

(b)  $S^b = \{s \in \mathcal{S} \times \mathcal{S}, s_2 > s_1 \neq 0\}$  :

$$h(y' | y, S^b) \geq \left( \frac{K^2+1}{2} + K-1 \right) \sum_{s \in S^b} \frac{p_s}{K^2}.$$

(c)  $S^c = \{s \in \mathcal{S} \times \mathcal{S}, s_1 = 2s_2 \neq 0\}$  :

$$h(y' | y, S^c) \geq \left( \frac{K^2+1}{2} + \frac{K-2}{2} \right) \sum_{s \in S^c} \frac{p_s}{K^2}.$$

(d)  $S^d = \{s \in \mathcal{S} \times \mathcal{S}, s_1 > s_2 \neq 0, \text{ and } s_1 \neq 2s_2\}$  :

$$h(y' | y, S^d) \geq \frac{K^2+1}{2} \sum_{s \in S^d} \frac{p_s}{K^2}$$

(e)  $S^e = \{s \in \mathcal{S} \times \mathcal{S}, s = (0, s_2) \text{ and } s_2 \neq 0\}$  :

$$h(y' | y, S^e) \geq \frac{K(K+1)}{2} \sum_{s \in S^e} \frac{p_s}{K^2}.$$

(f)  $S^f = \{s \in \mathcal{S} \times \mathcal{S}, s = (s_1, 0) \text{ and } s_1 \neq 0\}$  :

$$h(y' | y, S^f) \geq \frac{K(K-1)}{2} \sum_{s \in S^f} \frac{p_s}{K^2}.$$

(g)  $S^g = \{s \in \mathcal{S} \times \mathcal{S}, s = (0, 0)\}$  :

$$h(y' | y, S^g) = \frac{K^2}{2} \sum_{s \in S^g} \frac{p_s}{K^2}.$$

Pooling all types of voters, it can be easily shown that: (i) for all  $z$  with  $z_1 < y_1$  we have

$$h(z | y) < h(y' | y);$$

and (ii)

$$h(y' | y) = \sum_{i \in \{a, b, c, d, e, f, g\}} h(y' | y, S^i).$$

Using equalities  $\sum_{s \in \cup_{i \in I} S^i} p_s = 1$ ,  $\sum_{s \in S^b} p_s = \sum_{s \in S^c} p_s + \sum_{s \in S^d} p_s$ , and  $\sum_{s \in S^e} p_s = \sum_{s \in S^f} p_s$  in manipulations, we have

$$h(y' | y) = \frac{1}{2} + \frac{1}{K^2} \left( (K-1) \sum_{s \in S^a} p_s + \left( \frac{3}{2}K - 1 \right) \sum_{s \in S^c} p_s + K \sum_{s \in S^d} p_s \right)$$

Using (9), a sufficient condition for  $h(y' | y) > h(C | y)$  is given by

$$\frac{1}{2} + \frac{1}{K^2} \left( (K-1) \sum_{s \in S^a} p_s + \left( \frac{3}{2}K - 1 \right) \sum_{s \in S^c} p_s \right) > \frac{1}{2} + \frac{(1-p_0)}{2K}$$

Substituting in the above inequality  $\sum_{s \in S^a} p_s = \sum_{s \in S \setminus \{0\}} p_s^2$ ,  $\sum_{s \in S^c} p_s = \sum_{(s_1/s_2)=2} p_{s_1} p_{s_2}$ , and  $1 - p_0 = \sum_{s \in S \setminus \{0\}} p_s$ , we derive

$$\frac{(1 - \frac{1}{K}) \sum_{s \in S \setminus \{0\}} p_s^2 + (\frac{3}{2} - \frac{1}{K}) \sum_{(s_1/s_2)=2} p_{s_1} p_{s_2}}{\sum_{s \in S \setminus \{0\}} p_s} > \frac{1}{2}. \quad (10)$$

Furthermore, for each  $S^i$ ,  $i \in \{a, b, c, d, g\}$ , for  $z_1 \neq y_1$ , we have

$$h(z | y, S^i) \leq h(y' | y, S^i),$$

with strict inequality for  $i = a$ , and

$$h(z | y, S^e) + h(z | y, S^f) \leq h(y' | y, S^e) + h(y' | y, S^f).$$

Aggregating over the different sets, we find

$$h(z | y) < h(y' | y).$$

For  $z_1 = y_1$  we have

$$h(z | y) < h(C | y) < h(y' | y).$$

**Proof of Proposition 5.** First, recall the definitions of a persistent and a transient state. A state  $i$  is *persistent* if a system starting at  $i$  is certain to return to  $i$ . The state  $i$  is *transient* in the opposite case (see Billingsley (1986, p.114)).

1.-2. Apply Proposition 3.

3. (a) Let the process start at  $U_1$  and the incumbent be at  $y \in U_1$ . First, it can be easily shown that if a voter  $j$  with ideal position  $x_j$  and a vector of strengths  $s$  joins the interest group on the first issue then so do all other voters with ideal positions  $x$ ,  $x \in \{x, x_1 = x_{j1}\}$ , and vector of strengths  $s$ . In words, we can say that voters with the same vector of strengths and the same ideal position on one issue are homogenous. Next, the uniformity distribution and the homogeneity property imply that the median position on issue  $i$  for members of the group on issue  $3-i$  is expected to be  $y'_i$ . Hence, under the conditional polling there is a poll run at  $y'$ . From Proposition 3.2, there exists a  $y' \in U_1$ , such that  $h(y' | y) > h(z | y)$ , for all  $z \notin U_1$ , which implies that the new policy outcome will be  $y' \in U_1$ . Thus,  $U_1$  is a persistent state and  $\{U_1\}$  is closed.

(b) Since  $y' \neq y$ , voting cycles are generated once the incumbent enters state  $U_1$ .

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