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James C. Cox, Vjollca Sadiraj, and Bodo Vogt

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Expected value theory has been known for centuries to be subject to critique by St. Petersburg paradox arguments. And there is a traditional rebuttal of the critique that denies the empirical relevance of the paradox because of its apparent dependence on existence of credible offers to pay unbounded sums of money. Neither critique nor rebuttal focus on the question with empirical relevance: Do people make choices in bounded St. Petersburg games that are consistent with expected value theory? This paper reports an experiment that addresses that question.

Keywords: St. Petersburg paradox, expected value theory, experiment

1. Introduction

The first theory of decision under risk, expected value maximization, was challenged long ago by the St. Petersburg paradox (Bernoulli, 1738). The original St. Petersburg lottery pays 2^n when a fair coin comes up heads for the first time on flip n, an event with probability $1/2^n$. The expected value of this lottery is infinite: $\sum_{n=1}^{\infty} 2^n \times (1/2^n) = \infty$. But Bernoulli famously reported that most people stated they would be unwilling to pay more than a small amount to play this game. He concluded that such reported preferences called into question the validity of expected value theory, and offered a theory with decreasing marginal utility of money as a replacement.

A traditional rebuttal of the alleged paradox is based on the observation that no agent can credibly offer the St. Petersburg lottery for another to play in a real payoff experiment – because it could result in a payout obligation exceeding any agent's wealth – and therefore that this challenge to expected value theory has no bite. For example, if the maximum amount that a supplier of the game can credibly offer to pay is $\$3.3554\times10^7$ $(=\$2^{25})$ then the St. Petersburg lottery is a game that *actually pays* $\$\,2^n$ if $n<25$, and $\$\,2^{25}$ for $n\geq 25$. The expected value of this game is only \$26, so it would not be paradoxical if individuals stated they would be unwilling to pay large amounts to play the game.

Arguments about the relevance of the St. Petersburg paradox are great sport, especially now that a generalized form of the paradox has been shown (by Cox and Sadiraj, 2008 and Rieger and Wang, 2006) to apply to cumulative prospect theory (Tversky and Kahneman, 1992), dual theory of expected utility theory (Yaari, 1987), rank dependent utility theory (Quiggin, 1993), and expected utility theory as well as expected value theory. But such arguments are not an answer to the original question: Does expected value theory have empirical validity? This question has previously been addressed by numerous empirical studies but, surprisingly, not by experiments involving St. Petersburg lotteries. This paper reports an experiment with bounded St. Petersburg lotteries.

2. A Real Experiment with a Finite St. Petersburg Lottery

The experiment was designed as follows. Subjects were offered the opportunity to decide whether to pay their own money to play nine finite St. Petersburg bets. One of each subject's decisions was randomly selected for real money payoff. Bet *N* had a maximum of *N* coin tosses and paid 2^n euros if the first head occurred on toss number n, for $n=1,2,...N,$ and paid nothing if no head occurred. Bets were offered for *N* = 1,2, …,9. Of course, the expected payoff from playing bet *N* was *N* euros. The price offered to a subject for playing bet *N* was

25 euro cents lower than N euros. An expected value maximizer would accept every one of these bets.

The experiment was run at the University of Magdeburg in February 2007. An English version of the subject instructions is contained in the appendix. Thirty subjects participated in this experiment: (a) 127 out of 270 (or 47%) of the subjects' choices are inconsistent with risk neutrality; and (b) 26 out of 30 (or 87%) of the subjects made at least one choice inconsistent with risk neutrality. Is the failure rate of expected value theory statistically significant?

One way to pose the question is to ask which characterization of risk references is more consistent with the data: (a) risk neutrality; or (b) risk aversion sufficient to imply rejection of all offers to play the St. Petersburg games in the experiment? First, the observed fraction of choices consistent with risk neutrality is [0.87, 0.87, 0.8, 0.63, 0.53, 0.43, 0.27, 0.2, 0.17] in St. Petersburg bets for N = 1,2, …, 9, respectively. Recall that expected values of the nine St. Petersburg bets are [1, 2, 3, 4, 5, 6, 7, 8, 9], respectively. So in the first 5 tasks more than half the subjects made choices that are consistent with expected value (EV) theory. However, as the stakes of the sure amount of money required for playing the St. Petersburg bets increase, and the variance of payoffs of the lotteries increase, subjects risk-averse choices start to dominate. From bets for N = 6 to 9, the fraction of choices violating EV theory increases from 57% to 83%.

We apply a linear mixture model (Harless and Camerer, 1994) with stochastic preference specification for error rate ε: (a) if option Z is stochastically preferred then Prob(choose Z) $=1-\varepsilon$; and (b) if option Z is not preferred then Prob(choose Z) $=\varepsilon$. Let the letter a denote a subject's response that she accepts the offer to play a specific St. Petersburg bet in the experiment. Let r denote rejection of the offer to play the game. The linear mixture model is used to address the specific question whether, for the nine St. Petersburg bets offered to the subjects, the response pattern (a,a,a,a,a,a,a,a,a) or the response pattern (r,r,r,r,r,r,r,r,r) is more consistent with the data. With this specification, the log-likelihood is -182 and the estimate of the error rate is 0.29. The point estimate of the proportion of subjects in the experiment that are not risk neutral (or very slightly risk averse) is 0.48, and the Wald 90% confidence interval of this estimate is (0.30, 0.67). Allowing for a specification with two error rates (one error rate for bets 1-4 and another error rate for bets 5-9), the estimates of the error rates are 0.50 for the first four bets and 0.14 for the last five bets. The point estimate of the proportion of subjects in the experiment that are not risk neutral (or very slightly risk averse) is 0.77, with Wald 90% confidence interval of (0.63, 0.91). The log-likelihood is -164. Using data only for bets 4 - 9, that require payments in excess of 3 euros to accept, the error rate is 0.18 and the point estimate of the proportion of subjects in the experiment that are not risk neutral (or very slightly risk averse) is 0.71 with 90% confidence interval (0.56, 0.87). The log-likelihood is -103. We conclude that data from the St. Petersburg experiment are significantly inconsistent with expected value theory.

Endnote

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Subject Instructions

<u>Please write your identification code here</u>:

A coin is tossed no more than 9 times. Your payoff depends on the number of tosses until Head appears for the first time. If Head appears for the first time on flip number N then you are paid 2^N Euros. Table 1 shows the possible outcomes:

Table 1

Head appears for the first time on coin toss	Probability this will occur	Your payoff in Euros
1	0.5	2
2	0.25	4
3	0.125	8
4	0.0625	16
5	0.03125	32
6	0.015625	64
7	0.0078125	128
8	0.0039062	256
9	0.0019531	512
Never		0

There are a variety of different lotteries offered to you that differ in the maximum possible number of coin tosses and the amount you have to pay if you want to participate in the lottery. Table 2 shows this.

Table 2

Maximum number of tosses	Participation fee in Euros	I choose to pay to participate: Yes/No
1	0.75	
2	1.75	
3	2.75	
4	3.75	
5	4.75	
6	5.75	
7	6.75	
8	7.75	
9	8.75	

For example, if you decide to pay 3.75 Euros to participate in the lottery with a maximum of 4 tosses, the coin will be flipped 4 times. Your payoff is determined according to Table 1. If Head appears on the first toss then you will receive 2 Euros, regardless of the results of the further tosses. If Tail appears on the first toss and Head on the second, you will receive 4 Euros, regardless the results of the further tosses. If Tails appear on the first two tosses and then Head on the third toss, you will receive 8 Euros, regardless of the further tosses. If Tails appear on the first three tosses, followed by Head on the fourth toss, you receive 16 Euros. If Head never appears your payoff is 0 Euro.

Payoffs

After you make your decisions, one of the rows will be selected by chance and your Yes or No decision in that row will become binding. The selection of the row is carried out by drawing a ball from a bingo cage containing balls with numbers 1,2, …, 9. The number on the drawn ball determines the row of the table that is selected.

If, for example, row 6 is selected for payoff then: (a) if your decision in row 6 is “No” then no money changes hands; (b) if your decision in row 6 is “Yes” then you will pay the experimenter 5.75 Euros to play the coin toss lottery with a maximum of 6 tosses and possible outcomes in the first 6 rows of Table 1.